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The Expected Value of a Generalized Rayleigh Ratio, with Circuits and Antennas in Mind

FRÉDÉRIC BROYDÉ[1], and
EVELYNE CLAVELIER[2]

[1]Eurexcem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France
[2]Excem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France
Corresponding author: Frédéric Broydé (e-mail: fredbroyde@eurexcem.com).

ABSTRACT Generalized Rayleigh ratios play an important role in electrical engineering, notably in the theory of the power gains of a multiport device, and the theory of the gains of a multiport antenna array. For a generalized Rayleigh ratio of a first matrix (an hermitian matrix) to a second matrix (a positive definite matrix of same size as the first one) and a specified probability density of a normalized random excitation, we study the expected value of the generalized Rayleigh ratio. We compute a "surface-element-ratio probability density" that only depends on the second matrix and is such that the expected value of the generalized Rayleigh ratio is equal to the trace of the product of the first matrix and the inverse of the second matrix, divided by the number of rows (or columns) of these matrices.

INDEX TERMS Generalized Rayleigh ratio, transducer power gain, insertion power gain, power transfer ratio, operating power gain, available power gain, unnamed power gain, absolute gain, reached gain.

I. INTRODUCTION

Let a passive linear time-invariant (LTI) multiport device operate in the harmonic steady state. A number ν of its ports are coupled to a multiport generator, and the remaining ports are coupled to a multiport load. The transducer power gain, insertion power gain, power transfer ratios, operating power gain, available power gain and unnamed power gain of the multiport device are generalized Rayleigh ratios of a complex column vector [1]–[3]. This column vector is of size ν and represents the excitation of the multiport device.

If an LTI multiport antenna array (MAA) is used for emission, its partial absolute gain in any given direction and wave polarization, absolute gain in any given direction, partial reached gain in any given direction and wave polarization, and reached gain in any given direction are generalized Rayleigh ratios of a complex column vector [4]. This column vector represents the excitation of the MAA. It is of size ν, where ν is the number of ports of the MAA, all of which being coupled to a multiport generator during emission.

An important property of each of the generalized Rayleigh ratios mentioned above is its maximum and minimum values for all nonzero excitations lying in $\mathbb{C}^{\nu}$. These maximum and minimum values correspond to the largest and least eigenvalues of a matrix, respectively, and they have remarkable properties when suitable conditions are satisfied, at least one of them involving reciprocity. The trace of this matrix also has remarkable properties when suitable conditions are satisfied, at least one of them involving reciprocity [1]–[4].

However, the physical significance of this trace has not yet been investigated in detail.

The main purpose of this article is to provide a probabilistic interpretation of this trace, as the expected value of the relevant generalized Rayleigh ratio, for a probability density to be determined. Moreover, for applications to the partial absolute gain, absolute gain, partial reached gain or reached gain of a MAA, it is desirable that this probability density only depends on the denominator of the generalized Rayleigh ratio, because this denominator is independent of the direction in which these gains are considered.

The article is organized as follows. Our notations and some definitions are provided in Section II. Generalized Rayleigh ratios are briefly introduced in Section III. Section IV is used to study the definition and the computation of the surface element of the unit hypersphere of $\mathbb{C}^{\nu}$. Random excitations and the expected value of a generalized Rayleigh ratio for a specified probability density of a normalized random excitation are introduced and investigated in Section V. In Section VI, a "surface-element-ratio probability density" having the desired properties is presented. Sections VII and VIII provide examples.

II. NOTATIONS AND DEFINITIONS

We use $\mathbb{R}_+^*$ to denote the set of the positive real numbers. If $z \in \mathbb{C}$, the real part of z is denoted by $\mathrm{Re}(z)$, the imaginary part of z by $\mathrm{Im}(z)$, and the complex conjugate of z by $\bar{z}$.

If $\mathbf{M}$ is a complex matrix, then: $\mathrm{Re}(\mathbf{M})$ is the real part of $\mathbf{M}$, that is, the real matrix each entry of which is the real part of the corresponding entry of $\mathbf{M}$; and $\mathrm{Im}(\mathbf{M})$ is the imaginary part of $\mathbf{M}$, that is, the real matrix each entry of which is the imaginary part of the corresponding entry of $\mathbf{M}$.

Let $\mathbf{M}$ be a complex matrix. We use $\ker \mathbf{M}$ to denote the nullspace of $\mathbf{M}$, $\mathrm{rank}\, \mathbf{M}$ the rank of $\mathbf{M}$, $\mathbf{M}^{\mathrm{T}}$ the transpose of $\mathbf{M}$, and $\mathbf{M}^*$ the hermitian adjoint of $\mathbf{M}$. We use $\overline{\mathbf{M}}$ to denote the complex conjugate of $\mathbf{M}$, so that $\mathbf{M}^* = \overline{\mathbf{M}}^{\mathrm{T}}$. If $\mathbf{M}$ is square, $\mathrm{tr}\, \mathbf{M}$ denotes the trace of $\mathbf{M}$, $\det \mathbf{M}$ the determinant of $\mathbf{M}$, and $H(\mathbf{M})$ the hermitian part of $\mathbf{M}$ given by

$$H(\mathbf{M}) = \frac{\mathbf{M} + \mathbf{M}^*}{2} . \tag{1}$$

Let $\mathbf{A}$ be a positive semidefinite matrix. We know [5, Sec. 7.2.6] that there exists a unique positive semidefinite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. The matrix $\mathbf{B}$ is referred to as the unique positive semidefinite square root of $\mathbf{A}$, and is denoted by $\mathbf{A}^{1/2}$. If $\mathbf{A}$ is positive definite, $\mathbf{A}^{-1}$ and $\mathbf{A}^{1/2}$ are positive definite, and $(\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$, so that we can write $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$.

Let ν be a positive integer. We use $\mathbf{1}_\nu$ to denote the identity matrix of size ν by ν. The complex vector space of the complex column vectors of size ν is denoted by $\mathbb{C}^\nu$, and the real vector space of the real column vectors of size ν is denoted by $\mathbb{R}^\nu$. $\mathbb{C}^\nu$ is also a real vector space, of size 2ν.

We use $\langle \cdot, \cdot \rangle$ to denote the inner product of $\mathbb{C}^\nu$ such that, for any $\mathbf{x}$ and $\mathbf{y}$ lying in $\mathbb{C}^\nu$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{x}$. We use $|| \cdot ||_2$ to denote the euclidian norm of $\mathbb{C}^\nu$ such that, for any $\mathbf{x} \in \mathbb{C}^\nu$, we have

$$|| \mathbf{x} ||_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} . \tag{2}$$

We use the dot product notation to denote the scalar product of $\mathbb{R}^\nu$ such that, for any $\mathbf{x}$ and $\mathbf{y}$ lying in $\mathbb{R}^\nu$, $\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^{\mathrm{T}} \mathbf{x}$. Since no confusion may arise, we use $|| \cdot ||_2$ to denote the euclidian norm of $\mathbb{R}^\nu$ such that, for any $\mathbf{x} \in \mathbb{R}^\nu$, we have

$$|| \mathbf{x} ||_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}} . \tag{3}$$

Let $\Psi_\nu : \mathbb{C}^\nu \to \mathbb{R}^{2\nu}$ be the map such that, for any $\mathbf{x} \in \mathbb{C}^\nu$,

$$\Psi_\nu(\mathbf{x}) = \begin{pmatrix} \mathrm{Re}(\mathbf{x}) \\ \mathrm{Im}(\mathbf{x}) \end{pmatrix} . \tag{4}$$

Regarding $\mathbb{C}^\nu$ as a vector space over $\mathbb{R}$, we find that Ψ_ν is a bijective linear map. For any $\mathbf{x}$ and $\mathbf{y}$ lying in $\mathbb{C}^\nu$ we have

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle = \big(\mathrm{Re}(\mathbf{y})^{\mathrm{T}} - j\mathrm{Im}(\mathbf{y})^{\mathrm{T}}\big)\big(\mathrm{Re}(\mathbf{x}) + j\mathrm{Im}(\mathbf{x})\big) = \\ \mathrm{Re}(\mathbf{y})^{\mathrm{T}}\, \mathrm{Re}(\mathbf{x}) + \mathrm{Im}(\mathbf{y})^{\mathrm{T}}\, \mathrm{Im}(\mathbf{x}) \\ + j\big(\mathrm{Re}(\mathbf{y})^{\mathrm{T}}\, \mathrm{Im}(\mathbf{x}) - \mathrm{Im}(\mathbf{y})^{\mathrm{T}}\, \mathrm{Re}(\mathbf{x})\big) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Psi_\nu(\mathbf{x}) \cdot \Psi_\nu(\mathbf{y}) = \big(\mathrm{Re}(\mathbf{y})^{\mathrm{T}} \quad \mathrm{Im}(\mathbf{y})^{\mathrm{T}}\big) \begin{pmatrix} \mathrm{Re}(\mathbf{x}) \\ \mathrm{Im}(\mathbf{x}) \end{pmatrix} = \\ \mathrm{Re}(\mathbf{y})^{\mathrm{T}} \mathrm{Re}(\mathbf{x}) + \mathrm{Im}(\mathbf{y})^{\mathrm{T}} \mathrm{Im}(\mathbf{x}) . \end{aligned} \tag{6}$$

It follows that $\Psi_\nu(\mathbf{x}) \cdot \Psi_\nu(\mathbf{y})$ need not be equal to $\langle \mathbf{x}, \mathbf{y} \rangle$. However, if we assume $\mathbf{y} = \mathbf{x}$ in (5)-(6), we get

$$||\Psi_\nu(\mathbf{x})||_2 = ||\mathbf{x}||_2 , \tag{7}$$

so that Ψ_ν is an isometry from the normed vector space $\mathbb{C}^\nu$ equipped with the norm defined by (2) to the normed vector space $\mathbb{R}^{2\nu}$ equipped with the norm defined by (3). Loosely speaking, we can say that this isometry does not preserve the inner product and scalar product used to define the norms (even though some textbooks may seem to contradict this).

Let $\mathbb{S}_\nu$ denote the hypersphere of the unit vectors of $\mathbb{C}^\nu$, that is, the unit hypersphere of $\mathbb{C}^\nu$. It follows from (7) that $\Psi_\nu(\mathbb{S}_\nu)$ is the unit hypersphere of $\mathbb{R}^{2\nu}$.

III. GENERALIZED RAYLEIGH RATIO

Let $\mathbf{A}$ be a hermitian matrix of size ν by ν. The expression $(\mathbf{x}^* \mathbf{A} \mathbf{x})/(\mathbf{x}^* \mathbf{x})$, where $\mathbf{x} \in \mathbb{C}^\nu$, is known as a Rayleigh ratio, or Rayleigh-Ritz ratio, or Rayleigh quotient [5, Sec. 4.2], [6, Sec. 4.2]. In this article, this concept is extended as follows. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive semidefinite. The generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$ is a real-valued function $r : \mathbb{C}^\nu \to \mathbb{R}$ such that, for any $\mathbf{x} \in \mathbb{C}^\nu$ satisfying $\mathbf{x}^* \mathbf{D} \mathbf{x} \neq 0$, we have

$$r(\mathbf{x}) = \frac{\mathbf{x}^* \mathbf{N} \mathbf{x}}{\mathbf{x}^* \mathbf{D} \mathbf{x}} . \tag{8}$$

Let $\mathbf{A}$ be a positive semidefinite matrix of size ν by ν. We know [5, Sec. 7.1.6] that, for any $\mathbf{x} \in \mathbb{C}^\nu$, $\mathbf{x}^* \mathbf{A}\, \mathbf{x} = 0$ if and only if $\mathbf{x} \in \ker \mathbf{A}$. In this article, we further assume that $\mathbf{D}$ is positive definite, so that $r(\mathbf{x})$ is defined for any $\mathbf{x} \neq \mathbf{0}$.

We will refer to $\mathbf{x}$ as the "excitation". It follows from (8) that, for $\mathbf{x} \neq \mathbf{0}$ and a fixed $\mathbf{x}/||\mathbf{x}||_2$, $r(\mathbf{x})$ does not depend on $||\mathbf{x}||_2$. Thus, the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \neq \mathbf{0}$ is equal to the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \in \mathbb{S}_\nu$.

Theorem [3, Sec. II], [7, Sec. II]. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. We define

$$\mathbf{M} = \mathbf{D}^{-1/2} \mathbf{N} \mathbf{D}^{-1/2} . \tag{9}$$

$\mathbf{M}$ is of size ν by ν, and hermitian. Thus, its eigenvalues are real. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{M}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{M}$. For any $\mathbf{x} \in \mathbb{C}^\nu$ satisfying $\mathbf{x} \neq \mathbf{0}$, we have

$$\lambda_{\min} = \min_{\mathbf{y} \neq \mathbf{0}} \frac{\mathbf{y}^* \mathbf{M} \mathbf{y}}{\mathbf{y}^* \mathbf{y}} \leqslant r(\mathbf{x}) \leqslant \lambda_{\max} = \max_{\mathbf{y} \neq \mathbf{0}} \frac{\mathbf{y}^* \mathbf{M} \mathbf{y}}{\mathbf{y}^* \mathbf{y}} . \tag{10}$$

Moreover,

- the equality $r(\mathbf{x}) = \lambda_{\max}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2} \mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\max}$;
- the equality $r(\mathbf{x}) = \lambda_{\min}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2} \mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\min}$; and
- $\mathbf{M}$ and $\mathbf{N}\mathbf{D}^{-1}$ are similar, so that the eigenvalues of $\mathbf{N}\mathbf{D}^{-1}$ are real, $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ and $\lambda_{\min}$ is the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$.

IV. INVESTIGATION OF $\mathbb{S}_\nu$

A. GENERAL CONSIDERATIONS

Before considering random excitations in Section V, we need to study $\mathbb{S}_\nu$, where ν is a positive integer.

In what follows, $\hat{\mathbf{x}}$ denotes an arbitrary element of $\mathbb{S}_\nu$, hence a unit complex vector of size ν, and $\breve{\mathbf{x}}$ denotes an arbitrary element of the unit hypersphere of $\mathbb{R}^{2\nu}$, hence a unit real vector of size 2ν.

We are going to define $2\nu - 1$ real parameters ζ_1 to $\zeta_{2\nu-1}$ that form a convenient parametrization of $\mathbb{S}_\nu$. Identifying $\mathbb{S}_\nu$ with the unit hypersphere of $\mathbb{R}^{2\nu}$ using the linear isometry Ψ_ν, the corresponding jacobian matrix $\mathbf{J}$ is a real matrix of size 2ν by $2\nu - 1$ such that, for any $q \in \{1, \ldots, 2\nu - 1\}$, column q of $\mathbf{J}$, denoted by $\mathbf{J}^{\langle q \rangle}$, is given by

$$\mathbf{J}^{\langle q \rangle} = \frac{\partial\, \Psi_\nu(\hat{\mathbf{x}})}{\partial \zeta_q} \,. \tag{11}$$

The surface element of $\mathbb{S}_\nu$ is [8, Ch. VI], [9, Sec. 2.10]:

$$dS_\nu = \sqrt{\det \mathbf{G}_\mathrm{M}}\; d\zeta_1 \ldots d\zeta_{2\nu-1} \,, \tag{12}$$

where the metric matrix

$$\mathbf{G}_\mathrm{M} = \mathbf{J}^\mathrm{T}\, \mathbf{J} \tag{13}$$

is of size $2\nu - 1$ by $2\nu - 1$.

B. PARAMETRIZATION AND SURFACE ELEMENT OF $\mathbb{S}_1$

An obvious parametrization of $\mathbb{S}_1$ uses 1 real parameter ϕ_0, in such a way that an arbitrary element of $\mathbb{S}_1$ is given by:

$$\hat{\mathbf{x}} = \begin{pmatrix} e^{j\phi_0} \end{pmatrix} \,, \tag{14}$$

where $\phi_0 \in [0, 2\pi)$. To define a surface element and a surface area of $\mathbb{S}_1$, we identify $\mathbb{S}_1$ with the unit hypersphere of $\mathbb{R}^2$, which is a circle of radius 1, using the isometric isomorphism $\Psi_1 : \mathbb{C} \to \mathbb{R}^2$, where $\mathbb{C}$ is regarded as a normed vector space over $\mathbb{R}$. Thus, a parametric equation of $\Psi_1(\mathbb{S}_1)$ is:

$$\breve{\mathbf{x}} = \Psi_1(\hat{\mathbf{x}}) = \begin{pmatrix} \mathrm{Re}(\hat{\mathbf{x}}) \\ \mathrm{Im}(\hat{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix} . \tag{15}$$

Here, $\zeta_1 = \phi_0$ and $\mathbf{J} = (-\sin\phi_0 \;\; \cos\phi_0)^\mathrm{T}$. It follows that $\mathbf{G}_\mathrm{M} = (1)$, so that the surface element of $\mathbb{S}_1$ is

$$dS_1 = d\phi_0 \,. \tag{16}$$

This leads us to the surface area of $\mathbb{S}_1$:

$$S_1 = \int_{\hat{\mathbf{x}} \in \mathbb{S}_1} dS_1 = 2\pi \,. \tag{17}$$

C. PARAMETRIZATION AND SURFACE ELEMENT OF $\mathbb{S}_2$

A simple parametrization of $\mathbb{S}_2$ uses 3 real parameters χ_1, ϕ_0 and ϕ_1, in such a way that an arbitrary element of $\mathbb{S}_2$ is given by

$$\hat{\mathbf{x}} = \begin{pmatrix} \sin\chi_1\; e^{j\phi_1} \\ \cos\chi_1\; e^{j\phi_0} \end{pmatrix} , \tag{18}$$

where $\chi_1 \in [0, \pi/2]$, $\phi_0 \in [0, 2\pi)$ and $\phi_1 \in [0, 2\pi)$. To define a surface element and a surface area of $\mathbb{S}_2$, we identify $\mathbb{S}_2$ with the unit hypersphere of $\mathbb{R}^4$, using the isometric isomorphism $\Psi_2 : \mathbb{C}^2 \to \mathbb{R}^4$, where $\mathbb{C}^2$ is regarded as a normed vector space over $\mathbb{R}$. It follows that a parametric equation of $\Psi_2(\mathbb{S}_2)$ is:

$$\breve{\mathbf{x}} = \Psi_2(\hat{\mathbf{x}}) = \begin{pmatrix} \mathrm{Re}(\hat{\mathbf{x}}) \\ \mathrm{Im}(\hat{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \sin\chi_1 \cos\phi_1 \\ \cos\chi_1 \cos\phi_0 \\ \sin\chi_1 \sin\phi_1 \\ \cos\chi_1 \sin\phi_0 \end{pmatrix} . \tag{19}$$

Here, $\zeta_1 = \chi_1$, $\zeta_2 = \phi_0$ and $\zeta_3 = \phi_1$. The matrices $\mathbf{J}$ and $\mathbf{G}_\mathrm{M}$ computed in the Appendix are such that the surface element of $\mathbb{S}_2$ is

$$dS_2 = \cos\chi_1 \sin\chi_1\; d\chi_1\, d\phi_0\, d\phi_1 \,. \tag{20}$$

This leads us to the surface area of $\mathbb{S}_2$:

$$S_2 = \iiint_{\hat{\mathbf{x}} \in \mathbb{S}_2} dS_2 = 2\pi^2 \,. \tag{21}$$

D. PARAMETRIZATION AND SURFACE ELEMENT OF $\mathbb{S}_3$

A simple parametrization of $\mathbb{S}_3$ uses 5 real parameters χ_1, χ_2, ϕ_0, ϕ_1 and ϕ_2, in such a way that an arbitrary element of $\mathbb{S}_3$ is given by

$$\hat{\mathbf{x}} = \begin{pmatrix} \sin\chi_1 \sin\chi_2\; e^{j\phi_2} \\ \sin\chi_1 \cos\chi_2\; e^{j\phi_1} \\ \cos\chi_1\; e^{j\phi_0} \end{pmatrix} , \tag{22}$$

in which $\chi_1 \in [0, \pi/2]$ and $\chi_2 \in [0, \pi/2]$, and in which $\phi_0 \in [0, 2\pi)$, $\phi_1 \in [0, 2\pi)$ and $\phi_2 \in [0, 2\pi)$.

To define a surface element and a surface area of $\mathbb{S}_3$, we identify $\mathbb{S}_3$ with the hypersphere of $\mathbb{R}^6$, using the isometric isomorphism $\Psi_3 : \mathbb{C}^3 \to \mathbb{R}^6$, where $\mathbb{C}^3$ is regarded as a normed vector space over $\mathbb{R}$. It follows that a parametric equation of $\Psi_3(\mathbb{S}_3)$ is:

$$\breve{\mathbf{x}} = \Psi_3(\hat{\mathbf{x}}) = \begin{pmatrix} \mathrm{Re}(\hat{\mathbf{x}}) \\ \mathrm{Im}(\hat{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \sin\chi_1 \sin\chi_2 \cos\phi_2 \\ \sin\chi_1 \cos\chi_2 \cos\phi_1 \\ \cos\chi_1 \cos\phi_0 \\ \sin\chi_1 \sin\chi_2 \sin\phi_2 \\ \sin\chi_1 \cos\chi_2 \sin\phi_1 \\ \cos\chi_1 \sin\phi_0 \end{pmatrix} . \tag{23}$$

Here, $\zeta_1 = \chi_1$, $\zeta_2 = \chi_2$, $\zeta_3 = \phi_0$, $\zeta_4 = \phi_1$ and $\zeta_5 = \phi_2$. The matrices $\mathbf{J}$ and $\mathbf{G}_\mathrm{M}$ computed in the Appendix are such that the surface element of $\mathbb{S}_3$ is

$$dS_3 = \cos\chi_1 \sin^3\!\chi_1 \cos\chi_2 \sin\chi_2 \times d\chi_1\, d\chi_2\, d\phi_0\, d\phi_1\, d\phi_2 \,. \tag{24}$$

This leads us to the surface area of $\mathbb{S}_3$:

$$S_3 = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_3} dS_3 = \pi^3 \,. \tag{25}$$

E. PARAMETRIZATION AND SURFACE ELEMENT OF $\mathbb{S}_\nu$

Let ν be an integer such that $\nu \geqslant 4$. A simple parametrization of $\mathbb{S}_\nu$ uses $2\nu-1$ real parameters χ_1 to $\chi_{\nu-1}$ and ϕ_0 to $\phi_{\nu-1}$, in such a way that an arbitrary element of $\mathbb{S}_\nu$ is given by

$$\hat{\mathbf{x}} = \begin{pmatrix} \sin\chi_1 \cdots \sin\chi_{\nu-1}\; e^{j\phi_{\nu-1}} \\ \sin\chi_1 \cdots \sin\chi_{\nu-2}\cos\chi_{\nu-1}\; e^{j\phi_{\nu-2}} \\ \vdots \\ \sin\chi_1 \cos\chi_2\; e^{j\phi_1} \\ \cos\chi_1\; e^{j\phi_0} \end{pmatrix}, \qquad (26)$$

where χ_1 to $\chi_{\nu-1}$ lie in $[0, \pi/2]$, and where ϕ_0 to $\phi_{\nu-1}$ lie in $[0, 2\pi)$. To define a surface element and a surface area of $\mathbb{S}_\nu$, we identify $\mathbb{S}_\nu$ with the unit hypersphere of $\mathbb{R}^{2\nu}$, using the isometric isomorphism $\Psi_\nu : \mathbb{C}^\nu \to \mathbb{R}^{2\nu}$, where $\mathbb{C}^\nu$ is regarded as a normed vector space over $\mathbb{R}$. It follows that a parametric equation of $\Psi_\nu(\mathbb{S}_\nu)$ is:

$$\breve{\mathbf{x}} = \Psi_\nu(\hat{\mathbf{x}}) = \begin{pmatrix} \mathrm{Re}(\hat{\mathbf{x}}) \\ \mathrm{Im}(\hat{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \sin\chi_1 \cdots \sin\chi_{\nu-1} \cos\phi_{\nu-1} \\ \sin\chi_1 \cdots \sin\chi_{\nu-2}\cos\chi_{\nu-1}\cos\phi_{\nu-2} \\ \vdots \\ \sin\chi_1\cos\chi_2\cos\phi_1 \\ \cos\chi_1\cos\phi_0 \\ \sin\chi_1 \cdots \sin\chi_{\nu-1}\sin\phi_{\nu-1} \\ \sin\chi_1 \cdots \sin\chi_{\nu-2}\cos\chi_{\nu-1}\sin\phi_{\nu-2} \\ \vdots \\ \sin\chi_1\cos\chi_2\sin\phi_1 \\ \cos\chi_1\sin\phi_0 \end{pmatrix}. \qquad (27)$$

In this case $\nu \geqslant 4$, we have $\zeta_1 = \chi_1, \ldots, \zeta_{\nu-1} = \chi_{\nu-1}$ and $\zeta_\nu = \phi_0, \ldots, \zeta_{2\nu-1} = \phi_{\nu-1}$. The matrices $\mathbf{J}$ and $\mathbf{G}_\mathrm{M}$ computed in the Appendix are such that the surface element of $\mathbb{S}_\nu$ is

$$dS_\nu = \cos\chi_1 \sin^{2\nu-3}\chi_1 \cos\chi_2 \sin^{2\nu-5}\chi_2 \cdots \times \cos\chi_{\nu-1}\sin\chi_{\nu-1}\, d\chi_1 \cdots d\chi_{\nu-1}\, d\phi_0 \cdots d\phi_{\nu-1}. \qquad (28)$$

This leads us to the surface area of $\mathbb{S}_\nu$:

$$S_\nu = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} dS_\nu = \frac{2\pi^\nu}{(\nu-1)!}. \qquad (29)$$

If we compare (29) to (17), (21) and (25), we find that (29) is in fact applicable to any positive ν.

Some authors use S_ν, and others $S_{\nu+1}$, to denote the surface area of a hypersphere of $\mathbb{R}^{\nu+1}$, of unit radius. According to our notations, S_ν is the surface area of $\mathbb{S}_\nu$, which is equal to the surface area of the unit hypersphere of $\mathbb{R}^{2\nu}$. Thus, (29) is a well known result [10, p. 877].

V. RANDOM EXCITATION

A. PROBABILITY DENSITY AND EXPECTED VALUE

Let ν be a positive integer, and $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, where $\mathbf{D}$ is positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, in the variable $\mathbf{x} \in \mathbb{C}^\nu$ that is referred to as "excitation".

As regards $r(\mathbf{x})$, a random nonzero $\mathbf{x} \in \mathbb{C}^\nu$ is fully characterized by the probability density f of the normalized random excitation $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2 \in \mathbb{S}_\nu$, because $r(\mathbf{x}) = r(\hat{\mathbf{x}})$. This probability density must satisfy the normalization condition

$$\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} f(\hat{\mathbf{x}})\, dS_\nu = 1. \qquad (30)$$

The expected value of the generalized Rayleigh ratio r for the probability density f is

$$\langle r(\hat{\mathbf{x}})\rangle_f = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} f(\hat{\mathbf{x}}) r(\hat{\mathbf{x}})\, dS_\nu = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} f(\hat{\mathbf{x}}) \frac{\hat{\mathbf{x}}^*\mathbf{N}\hat{\mathbf{x}}}{\hat{\mathbf{x}}^*\mathbf{D}\hat{\mathbf{x}}}\, dS_\nu. \qquad (31)$$

Using

$$\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}, \qquad (32)$$

we find

$$\langle r(\hat{\mathbf{x}})\rangle_f = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} f(\hat{\mathbf{x}}) \frac{(\mathbf{D}^{1/2}\hat{\mathbf{x}})^*\mathbf{M}(\mathbf{D}^{1/2}\hat{\mathbf{x}})}{(\mathbf{D}^{1/2}\hat{\mathbf{x}})^*(\mathbf{D}^{1/2}\hat{\mathbf{x}})}\, dS_\nu. \qquad (33)$$

$\mathbf{M}$ is of size ν by ν, and hermitian. It follows that its eigenvalues are real. Counting multiplicity, let us label them $\lambda_1, \ldots, \lambda_\nu$, and let $(\mathbf{u}_1, \ldots, \mathbf{u}_\nu)$ be an orthonormal basis of $\mathbb{C}^\nu$ such that for any $p \in \{1, \ldots, \nu\}$, $\mathbf{u}_p$ is an eigenvector of $\mathbf{M}$ associated with the eigenvalue λ_p. We obtain

$$\langle r(\hat{\mathbf{x}})\rangle_f = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} f(\hat{\mathbf{x}}) \frac{\sum_{p=1}^{\nu} \lambda_p\, |\langle \mathbf{D}^{1/2}\hat{\mathbf{x}}, \mathbf{u}_p\rangle|^2}{||\mathbf{D}^{1/2}\hat{\mathbf{x}}||_2^2}\, dS_\nu, \qquad (34)$$

where, for any $\mathbf{a}$ and $\mathbf{b}$ lying in $\mathbb{C}^\nu$, $\langle \mathbf{a}, \mathbf{b}\rangle = \mathbf{b}^*\mathbf{a}$, in line with the notations of Section II. This leads us to

$$\langle r(\hat{\mathbf{x}})\rangle_f = \sum_{p=1}^{\nu} \lambda_p \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} f(\hat{\mathbf{x}}) \left| \left\langle \frac{\mathbf{D}^{1/2}\hat{\mathbf{x}}}{||\mathbf{D}^{1/2}\hat{\mathbf{x}}||_2}, \mathbf{u}_p \right\rangle \right|^2 dS_\nu. \qquad (35)$$

B. FIRST PROBABILITY DENSITY

For any nonzero $\hat{\mathbf{a}} \in \mathbb{S}_\nu$, we use $\delta_\nu(\hat{\mathbf{x}} - \hat{\mathbf{a}})$ to denote the tempered distribution such that for any test function $\Phi : \mathbb{S}_\nu \to \mathbb{R}$, we have

$$\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} \delta_\nu(\hat{\mathbf{x}} - \hat{\mathbf{a}})\Phi(\hat{\mathbf{x}})\, dS_\nu = \Phi(\hat{\mathbf{a}}), \qquad (36)$$

where the left-hand side is not an actual integral but a notation meaning the image of Φ under $\delta_\nu(\hat{\mathbf{x}} - \hat{\mathbf{a}})$. Using again this notation, we get

$$\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} \left[\sum_{p=1}^{\nu} \delta_\nu \left(\hat{\mathbf{x}} - \frac{\mathbf{D}^{-1/2}\mathbf{u}_p}{||\mathbf{D}^{-1/2}\mathbf{u}_p||_2} \right) \right] dS_\nu = \nu. \qquad (37)$$

We can therefore assume that a random nonzero excitation $\mathbf{x} \in \mathbb{C}^\nu$ is such that the variable $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2 \in \mathbb{S}_\nu$ has a probability density

$$f_\mathrm{D} = \frac{1}{\nu} \sum_{p=1}^{\nu} \delta_\nu \left(\hat{\mathbf{x}} - \frac{\mathbf{D}^{-1/2}\mathbf{u}_p}{||\mathbf{D}^{-1/2}\mathbf{u}_p||_2} \right), \qquad (38)$$

because it follows from (37) that the tempered distribution $f_{\rm D}$ defined by (38) satisfies (30). This is the probability density of a discrete probability distribution according to which the possible events are $\mathbf{x}/||\mathbf{x}||_2 = \mathbf{v}_1, \ldots, \mathbf{x}/||\mathbf{x}||_2 = \mathbf{v}_\nu$, where, for any $p \in \{1, \ldots, \nu\}$,

$$\mathbf{v}_p = \frac{\mathbf{D}^{-1/2}\mathbf{u}_p}{||\mathbf{D}^{-1/2}\mathbf{u}_p||_2}\,, \tag{39}$$

and the probability of the event $\mathbf{x}/||\mathbf{x}||_2 = \mathbf{v}_p$ is $1/\nu$.

Utilizing (35), (36) and (38), we find that the expected value of the generalized Rayleigh ratio r for the probability density $f_{\rm D}$ is

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm D}} = \frac{1}{\nu}\sum_{p=1}^{\nu}\lambda_p = \frac{\operatorname{tr}\mathbf{M}}{\nu} = \frac{\operatorname{tr}(\mathbf{N}\mathbf{D}^{-1})}{\nu}\,. \tag{40}$$

C. A USEFUL INTEGRAL

To study the next two probability densities, we need to compute

$$I = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} |\langle\hat{\mathbf{x}},\mathbf{u}\rangle|^2\, dS_\nu \tag{41}$$

for an arbitrary fixed $\mathbf{u} \in \mathbb{S}_\nu$. Without loss of generality, we can assume that $\mathbf{u} = (0, \ldots, 0, 1)^T$. Using (26), we get

$$I = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} |\cos\chi_1 \exp(j\phi_0)|^2\, dS_\nu\,. \tag{42}$$

Using (28), we get

$$I = \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} \cos^3\chi_1 \sin^{2\nu-3}\chi_1 \cos\chi_2 \sin^{2\nu-5}\chi_2 \cdots$$
$$\times \cos\chi_{\nu-1}\sin\chi_{\nu-1}\, d\chi_1\cdots d\chi_{\nu-1}\, d\phi_0\cdots d\phi_{\nu-1}\,, \tag{43}$$

so that

$$I = \frac{(2\pi)^\nu}{2^{\nu-2}(\nu-2)!}\int_0^{\pi/2} \cos^3\chi_1 \sin^{2\nu-3}\chi_1\, d\chi_1 \tag{44}$$

and

$$I = \frac{4\pi^\nu}{(\nu-2)!}\int_0^{\pi/2} \cos\chi_1 \left(\sin^{2\nu-3}\chi_1 - \sin^{2\nu-1}\chi_1\right) d\chi_1\,. \tag{45}$$

We obtain

$$I = \frac{4\pi^\nu}{(\nu-2)!}\left[\frac{1}{2\nu-2} - \frac{1}{2\nu}\right] = \frac{2\pi^\nu}{\nu!}\,. \tag{46}$$

D. SECOND PROBABILITY DENSITY

We assume that a random nonzero excitation $\mathbf{x} \in \mathbb{C}^\nu$ is such that the variable $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2$ has a uniform probability density $f_{\rm U} : \mathbb{S}_\nu \to \mathbb{R}_+^*$, which neither depends on ϕ_0 to $\phi_{\nu-1}$, nor on χ_1 to $\chi_{\nu-1}$ if $\nu \geqslant 2$. It follows from (30) that, for any $\hat{\mathbf{x}} \in \mathbb{S}_\nu$,

$$f_{\rm U}(\hat{\mathbf{x}}) = \frac{1}{S_\nu}\,. \tag{47}$$

If we further assume that there exists $d \in \mathbb{R}_+^*$ such that

$$\mathbf{D} = d\,\mathbf{1}_\nu\,, \tag{48}$$

it follows from (35) and (47)–(48) that

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm U}} = \frac{1}{S_\nu}\sum_{p=1}^{\nu}\lambda_p \int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} |\langle\hat{\mathbf{x}},\mathbf{u}_p\rangle|^2\, dS_\nu\,. \tag{49}$$

Thus, using (29), (41), (46) and (49), we find that, if (48) is satisfied, then

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm U}} = \frac{1}{\nu}\sum_{p=1}^{\nu}\lambda_p = \frac{\operatorname{tr}\mathbf{M}}{\nu} = \frac{\operatorname{tr}\mathbf{N}}{d\nu}\,. \tag{50}$$

This expected value is the same as (40), though (40) and (50) are based on quite different assumptions.

E. THIRD PROBABILITY DENSITY

Let $\lambda_{\rm D1}, \ldots, \lambda_{{\rm D}\nu}$ be the real and positive eigenvalues of the positive definite matrix $\mathbf{D}$, counting multiplicity. Let $(\mathbf{u}_{\rm D1}, \ldots, \mathbf{u}_{{\rm D}\nu})$ be an orthonormal basis of $\mathbb{C}^\nu$ such that for any $p \in \{1, \ldots, \nu\}$, $\mathbf{u}_{{\rm D}p}$ is an eigenvector of $\mathbf{D}$ associated with the eigenvalue $\lambda_{{\rm D}p}$. Using (41) and (46), we obtain

$$\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} \hat{\mathbf{x}}^*\mathbf{D}\hat{\mathbf{x}}\; dS_\nu =$$
$$\sum_{p=1}^{\nu}\lambda_{{\rm D}p}\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} |\langle\hat{\mathbf{x}},\mathbf{u}_{{\rm D}p}\rangle|^2\, dS_\nu = \frac{2\pi^\nu}{\nu!}\operatorname{tr}\mathbf{D}\,. \tag{51}$$

We assume that a random nonzero excitation $\mathbf{x} \in \mathbb{C}^\nu$ is such that the variable $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2$ has a probability density $f_{\rm T} : \mathbb{S}_\nu \to \mathbb{R}_+^*$ given by

$$f_{\rm T}(\hat{\mathbf{x}}) = \frac{\nu}{S_\nu \operatorname{tr}\mathbf{D}}\,\hat{\mathbf{x}}^*\mathbf{D}\hat{\mathbf{x}}\,, \tag{52}$$

this choice being legitimate because it follows from (29) and (51) that $f_{\rm T}$ defined by (52) satisfies the normalization condition (30). We refer to $f_{\rm T}$ as the "denominator probability density" because it is the denominator $\hat{\mathbf{x}}^*\mathbf{D}\hat{\mathbf{x}}$ of $r(\hat{\mathbf{x}})$, suitably normalized. Using (29) and (52) in (31), we get

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm T}} = \frac{\nu!}{2\pi^\nu \operatorname{tr}\mathbf{D}}\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} \hat{\mathbf{x}}^*\mathbf{N}\hat{\mathbf{x}}\, dS_\nu\,. \tag{53}$$

Let $\lambda_{\rm N1}, \ldots, \lambda_{{\rm N}\nu}$ be the real and positive eigenvalues of the positive semidefinite matrix $\mathbf{N}$, counting multiplicity. Let $(\mathbf{u}_{\rm N1}, \ldots, \mathbf{u}_{{\rm N}\nu})$ be an orthonormal basis of $\mathbb{C}^\nu$ such that for any $p \in \{1, \ldots, \nu\}$, $\mathbf{u}_{{\rm N}p}$ is an eigenvector of $\mathbf{N}$ associated with the eigenvalue $\lambda_{{\rm N}p}$. We get

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm T}} = \frac{\nu!}{2\pi^\nu \operatorname{tr}\mathbf{D}}\sum_{p=1}^{\nu}\lambda_{{\rm N}p}\int\!\!\cdots\!\!\int_{\hat{\mathbf{x}}\in\mathbb{S}_\nu} |\langle\hat{\mathbf{x}},\mathbf{u}_{{\rm N}p}\rangle|^2\, dS_\nu\,. \tag{54}$$

Using (41) and (46), we obtain

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm T}} = \frac{\operatorname{tr}\mathbf{N}}{\operatorname{tr}\mathbf{D}}\,. \tag{55}$$

Note that, if $\mathbf{D} = d\,\mathbf{1}_\nu$, where $d \in \mathbb{R}_+^*$, it follows from (47) and (52) that $f_{\rm T} = f_{\rm U}$ and $\langle r(\hat{\mathbf{x}})\rangle_{f_{\rm T}} = \langle r(\hat{\mathbf{x}})\rangle_{f_{\rm U}}$.

VI. A WANTED PROBABILITY DENSITY

A. PRECURSORY OBSERVATIONS

For the reasons explained in Section I, we now look for a probability density f that: (a) only depends on $\mathbf{D}$; and (b) is such that $\langle r(\hat{\mathbf{x}})\rangle_f$ only depends on ν and $\operatorname{tr}\mathbf{M}$. Note that:

- f_D satisfies condition (b), but not condition (a) because the orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_\nu)$ depends on $\mathbf{N}$;
- f_U satisfies condition (a) but need not satisfy condition (b) if $\mathbf{D} \neq d\,\mathbf{1}_\nu$, in which $d \in \mathbb{R}_+^*$;
- f_T satisfies condition (a), but not condition (b) since we cannot say that $\langle r(\hat{\mathbf{x}})\rangle_{f_\mathrm{T}}$ only depends on $\operatorname{tr}\mathbf{M}$ and ν.

B. NEW PARAMETRIZATION OF $\mathbb{S}_\nu$

Let $\Xi : \mathbb{S}_\nu \to \mathbb{S}_\nu$ be such that, for any $\hat{\mathbf{x}} \in \mathbb{S}_\nu$,

$$\Xi(\hat{\mathbf{x}}) = \frac{\mathbf{D}^{1/2}\hat{\mathbf{x}}}{||\mathbf{D}^{1/2}\hat{\mathbf{x}}||_2}\,. \tag{56}$$

We easily check that Ξ is bijective and $\Xi^{-1} : \mathbb{S}_\nu \to \mathbb{S}_\nu$ is such that, for any $\hat{\mathbf{y}} \in \mathbb{S}_\nu$,

$$\Xi^{-1}(\hat{\mathbf{y}}) = \frac{\mathbf{D}^{-1/2}\hat{\mathbf{y}}}{||\mathbf{D}^{-1/2}\hat{\mathbf{y}}||_2}\,. \tag{57}$$

Since Ξ and Ξ^{-1} are continuously differentiable, Ξ is a diffeomorphism, which can be used to define a new parametrization of $\mathbb{S}_\nu$ as follows: an arbitrary element of $\mathbb{S}_\nu$ is given by $\hat{\mathbf{y}} = \Xi(\hat{\mathbf{x}})$ where $\hat{\mathbf{x}}$ is given by (14) if $\nu = 1$, or (18) if $\nu = 2$, or (22) if $\nu = 3$, or (26) if $\nu \geqslant 4$.

The corresponding jacobian matrix $\mathbf{J}'$ is a real matrix of size 2ν by $2\nu - 1$ such that, for any $q \in \{1, \ldots, 2\nu - 1\}$, column q of $\mathbf{J}'$, denoted by $\mathbf{J}'^{\langle q\rangle}$, is given by

$$\mathbf{J}'^{\langle q\rangle} = \frac{\partial\,\Psi_\nu(\hat{\mathbf{y}})}{\partial\zeta_q} = \frac{\partial\,\Psi_\nu(\Xi(\hat{\mathbf{x}}))}{\partial\zeta_q}\,. \tag{58}$$

It follows that the corresponding surface element of $\mathbb{S}_\nu$ is

$$dS'_\nu = \sqrt{\det \mathbf{G}'_\mathrm{M}}\; d\zeta_1 \ldots d\zeta_{2\nu-1}\,, \tag{59}$$

where the metric matrix

$$\mathbf{G}'_\mathrm{M} = \mathbf{J}'^\mathrm{T}\,\mathbf{J}' \tag{60}$$

is of size $2\nu - 1$ by $2\nu - 1$. The new parametrization of $\mathbb{S}_\nu$ is necessarily such that

$$\underset{\hat{\mathbf{x}}\in\mathbb{S}_\nu}{\int\!\cdots\!\int} dS'_\nu = S_\nu = \frac{2\pi^\nu}{(\nu-1)!}\,, \tag{61}$$

in which we have used (29) and the fact that, Ξ being a bijection, $\Xi(\hat{\mathbf{x}}) \in \mathbb{S}_\nu$ if and only if $\hat{\mathbf{x}} \in \mathbb{S}_\nu$.

C. COMPUTATION OF THE NEW JACOBIAN MATRIX

The matrix

$$\mathbf{U} = \begin{pmatrix} \operatorname{Re}(\mathbf{D}^{1/2}) & -\operatorname{Im}(\mathbf{D}^{1/2}) \\ \operatorname{Im}(\mathbf{D}^{1/2}) & \operatorname{Re}(\mathbf{D}^{1/2}) \end{pmatrix} \tag{62}$$

is symmetric because $\mathbf{D}^{1/2}$ is positive definite, and satisfies

$$\mathbf{U}\begin{pmatrix} \operatorname{Re}(\hat{\mathbf{x}}) \\ \operatorname{Im}(\hat{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(\mathbf{D}^{1/2}\hat{\mathbf{x}}) \\ \operatorname{Im}(\mathbf{D}^{1/2}\hat{\mathbf{x}}) \end{pmatrix}, \tag{63}$$

so that

$$\mathbf{U}\Psi_\nu(\hat{\mathbf{x}}) = \Psi_\nu(\mathbf{D}^{1/2}\hat{\mathbf{x}})\,. \tag{64}$$

It follows from (56) and (64) that

$$\Psi_\nu(\Xi(\hat{\mathbf{x}})) = \frac{\mathbf{U}\Psi_\nu(\hat{\mathbf{x}})}{||\mathbf{D}^{1/2}\hat{\mathbf{x}}||_2}\,. \tag{65}$$

Since Ψ_ν is an isometry, it follows from (64) that we have $||\mathbf{D}^{1/2}\hat{\mathbf{x}}||_2 = ||\mathbf{U}\Psi_\nu(\hat{\mathbf{x}})||_2$, so that (65) leads us to

$$\Psi_\nu(\Xi(\hat{\mathbf{x}})) = \frac{\mathbf{U}\breve{\mathbf{x}}}{||\mathbf{U}\breve{\mathbf{x}}||_2}\,, \tag{66}$$

where we have used $\breve{\mathbf{x}}$ to denote $\Psi_\nu(\hat{\mathbf{x}})$. Accordingly, for any $q \in \{1, \ldots, 2\nu - 1\}$, we have

$$\frac{\partial\Psi_\nu(\Xi(\hat{\mathbf{x}}))}{\partial\zeta_q} = \frac{\mathbf{U}}{||\mathbf{U}\breve{\mathbf{x}}||_2}\frac{\partial\breve{\mathbf{x}}}{\partial\zeta_q} - \frac{\mathbf{U}\breve{\mathbf{x}}}{||\mathbf{U}\breve{\mathbf{x}}||_2^2}\frac{\partial||\mathbf{U}\breve{\mathbf{x}}||_2}{\partial\zeta_q}\,. \tag{67}$$

Since $||\mathbf{U}\breve{\mathbf{x}}||_2 = \sqrt{\breve{\mathbf{x}}^\mathrm{T}\mathbf{U}^\mathrm{T}\mathbf{U}\breve{\mathbf{x}}}$, we have

$$\begin{aligned}\frac{\partial||\mathbf{U}\,\breve{\mathbf{x}}||_2}{\partial\zeta_q} &= \frac{1}{2||\mathbf{U}\,\breve{\mathbf{x}}||_2}\left[\left(\frac{\partial\breve{\mathbf{x}}}{\partial\zeta_q}\right)^\mathrm{T}\mathbf{U}^\mathrm{T}\mathbf{U}\breve{\mathbf{x}} + \breve{\mathbf{x}}^\mathrm{T}\mathbf{U}^\mathrm{T}\mathbf{U}\frac{\partial\breve{\mathbf{x}}}{\partial\zeta_q}\right] \\ &= \frac{1}{||\mathbf{U}\,\breve{\mathbf{x}}||_2}\breve{\mathbf{x}}^\mathrm{T}\mathbf{U}^\mathrm{T}\mathbf{U}\frac{\partial\breve{\mathbf{x}}}{\partial\zeta_q}\,.\end{aligned} \tag{68}$$

Using (68) in (67), we get

$$\frac{\partial\Psi_\nu(\Xi(\hat{\mathbf{x}}))}{\partial\zeta_q} = \frac{1}{||\mathbf{U}\breve{\mathbf{x}}||_2}\left(\mathbf{1}_{2\nu} - \frac{\mathbf{U}\breve{\mathbf{x}}\,\breve{\mathbf{x}}^\mathrm{T}\mathbf{U}^\mathrm{T}}{||\mathbf{U}\,\breve{\mathbf{x}}||_2^2}\right)\mathbf{U}\frac{\partial\breve{\mathbf{x}}}{\partial\zeta_q}\,. \tag{69}$$

Using (11), (58), (69) and $\mathbf{U}^\mathrm{T} = \mathbf{U}$, we finally obtain the jacobian matrix of the new parametrization:

$$\mathbf{J}' = \frac{1}{||\mathbf{U}\breve{\mathbf{x}}||_2}\left(\mathbf{1}_{2\nu} - \frac{\mathbf{U}\breve{\mathbf{x}}\,\breve{\mathbf{x}}^\mathrm{T}\mathbf{U}}{||\mathbf{U}\,\breve{\mathbf{x}}||_2^2}\right)\mathbf{U}\mathbf{J}\,. \tag{70}$$

D. DEFINITION OF THE PROBABILITY DENSITY

Let Ω be the set of the $\hat{\mathbf{x}} \in \mathbb{S}_\nu$ such that $\det \mathbf{G}_\mathrm{M} = 0$, and Ω' be the set of the $\hat{\mathbf{x}} \in \mathbb{S}_\nu$ such that $\det \mathbf{G}'_\mathrm{M} = 0$.

We have: $\det \mathbf{G}_\mathrm{M} = 0$ if and only if 0 is an eigenvalue of $\mathbf{G}_\mathrm{M}$. By (13) and [5, Sec. 2.6.3], $\det \mathbf{G}_\mathrm{M} = 0$ if and only if 0 is a singular value of $\mathbf{J}$. Thus, $\det \mathbf{G}_\mathrm{M} = 0$ if and only if $\operatorname{rank}\mathbf{J} \leqslant \nu - 2$, because $\mathbf{J}$ has $\nu - 1$ singular values counting multiplicity, and the rank of $\mathbf{J}$ is equal to the number of its nonzero singular values [5, Sec. 2.6.3]. Thus, using (70) and [5, Sec. 0.4.5], we find that, if $\det \mathbf{G}_\mathrm{M} = 0$, then $\operatorname{rank}\mathbf{J}' \leqslant \nu - 2$, so that $\operatorname{rank}\mathbf{G}' \leqslant \nu - 2$, so that $\det \mathbf{G}'_\mathrm{M} = 0$. Consequently,

$$\Omega \subset \Omega'\,. \tag{71}$$

We assume that a random nonzero excitation $\mathbf{x} \in \mathbb{C}^\nu$ is such that the variable $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2$ has a probability density $f_\mathrm{W} : \mathbb{S}_\nu \to \mathbb{R}_+^*$, which is, for any $\hat{\mathbf{x}} \in \mathbb{S}_\nu$, given by

$$f_\mathrm{W}(\hat{\mathbf{x}}) = \frac{1}{S_\nu}\frac{\sqrt{\det \mathbf{G}'_\mathrm{M}}}{\sqrt{\det \mathbf{G}_\mathrm{M}}} \tag{72}$$

if $\hat{\mathbf{x}} \notin \Omega$. By (71) the right-hand side of (72) is indeterminate of the type 0/0 if $\hat{\mathbf{x}} \in \Omega$. We posit $f_\mathrm{W}(\hat{\mathbf{x}}) = a(\hat{\mathbf{x}})$ if $\hat{\mathbf{x}} \in \Omega$, where $a : \Omega \to \mathbb{R}_+^*$ is arbitrary. Note that the value of $a(\hat{\mathbf{x}})$ is immaterial because $f_\mathrm{W}(\hat{\mathbf{x}})$ is only used multiplied by dS_ν, and by (12) the product $f_\mathrm{W}(\hat{\mathbf{x}})\,dS_\nu$ is zero for any $\hat{\mathbf{x}} \in \Omega$ and any value of $a(\hat{\mathbf{x}})$.

This choice of f_{W} is legitimate because it follows from (12), (59) and (61) that f_{W} defined by (72) if $\hat{\mathbf{x}} \notin \Omega$, or by $f_{\mathrm{W}}(\hat{\mathbf{x}}) = a(\hat{\mathbf{x}})$ if $\hat{\mathbf{x}} \in \Omega$, satisfies the normalization condition (30). Furthermore, it follows from (12), (71) and (72) that, for any $\hat{\mathbf{x}} \in \mathbb{S}_\nu$, we have

$$f_{\mathrm{W}}(\hat{\mathbf{x}})\, dS_\nu = \frac{1}{S_\nu} \sqrt{\det \mathbf{G}'_{\mathrm{M}}}\; d\zeta_1 \ldots d\zeta_{2\nu-1}\,, \tag{73}$$

where the problem arising in (72) for $\hat{\mathbf{x}} \in \Omega$ does not occur.

By (33) and (56), we have

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} = \underset{\hat{\mathbf{x}} \in \mathbb{S}_\nu}{\int \!\!\cdots\!\! \int} f_{\mathrm{W}}(\hat{\mathbf{x}}) \frac{\Xi(\hat{\mathbf{x}})^* \mathbf{M}\, \Xi(\hat{\mathbf{x}})}{\Xi(\hat{\mathbf{x}})^*\, \Xi(\hat{\mathbf{x}})}\, dS_\nu\,. \tag{74}$$

It follows from (12), (59), (72) and (74) that

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} = \frac{1}{S_\nu} \underset{\hat{\mathbf{y}} \in \mathbb{S}_\nu}{\int \!\!\cdots\!\! \int} \hat{\mathbf{y}}^* \mathbf{M} \hat{\mathbf{y}}\, dS'_\nu\,, \tag{75}$$

where $\hat{\mathbf{y}} = \Xi(\hat{\mathbf{x}})$. This integral uses the new parametrization of $\mathbb{S}_\nu$, in which the surface element is dS'_ν and the variable is $\hat{\mathbf{y}} = \Xi(\hat{\mathbf{x}})$. It can be translated into an integral using the parametrization of $\mathbb{S}_\nu$ defined in Section IV, in which the surface element is dS_ν and the variable is $\hat{\mathbf{x}}$. Accordingly, we get

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} = \frac{1}{S_\nu} \underset{\hat{\mathbf{x}} \in \mathbb{S}_\nu}{\int \!\!\cdots\!\! \int} \hat{\mathbf{x}}^* \mathbf{M} \hat{\mathbf{x}}\, dS_\nu \tag{76}$$

and

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} = \frac{1}{S_\nu} \sum_{p=1}^{\nu} \lambda_p \underset{\hat{\mathbf{x}} \in \mathbb{S}_\nu}{\int \!\!\cdots\!\! \int} |\langle \hat{\mathbf{x}}, \mathbf{u}_p \rangle|^2\, dS_\nu\,. \tag{77}$$

Thus, using (29), (41) and (46), we obtain

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} = \frac{\operatorname{tr} \mathbf{M}}{\nu} = \frac{\operatorname{tr}\,(\mathbf{N}\mathbf{D}^{-1})}{\nu}\,. \tag{78}$$

If $\mathbf{D} = d\,\mathbf{1}_\nu$, where $d \in \mathbb{R}_+^*$, it follows from (56) that Ξ is an identity function, from (11) and (58) that $\mathbf{J}' = \mathbf{J}$, from (13) and (60) that $\mathbf{G}'_{\mathrm{M}} = \mathbf{G}_{\mathrm{M}}$, and from (47) and (72) that $f_{\mathrm{W}} = f_{\mathrm{U}}$ and $\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} = \langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{U}}}$. Note that, in this case $\mathbf{D} = d\,\mathbf{1}_\nu$, we find that $\mathbf{J}' = \mathbf{J}$ is a simple consequence of (11), (56) and (58), but not an obvious consequence of (69), unless: we notice that $\mathbf{U} = \sqrt{d}\,\mathbf{1}_{2\nu}$ when $\mathbf{D} = d\,\mathbf{1}_\nu$; and we see that, for any $q \in \{1, \ldots, 2\nu - 1\}$, $\breve{\mathbf{x}}$ is orthogonal to $\partial \breve{\mathbf{x}} / \partial \zeta_q$ because $||\breve{\mathbf{x}}||_2 = 1$.

E. DISCUSSION AND NAMING

By (11), (13), (58), (60), (72) and (78), f_{W} satisfies conditions (a) and (b) of Section VI.A. In other words, f_{W} has the wanted characteristics. To clarify the meaning of f_{W}, we can use (59) and (73) to obtain

$$f_{\mathrm{W}}(\hat{\mathbf{x}})\, dS_\nu = \frac{1}{S_\nu}\, dS'_\nu\,. \tag{79}$$

This entails that the probability density f_{W} in the variable $\hat{\mathbf{x}}$ corresponds to a uniform probability density in the variable $\hat{\mathbf{y}} = \Xi(\hat{\mathbf{x}})$. Furthermore, (79) also entails that $f_{\mathrm{W}}(\hat{\mathbf{x}})$ may be viewed as the ratio of surface elements dS'_ν / dS_ν, divided by S_ν. For this reason, we refer to f_{W} as the "surface-element-ratio probability density".

VII. FIRST EXAMPLE

In a first example, $\nu = 2$,

$$\mathbf{N} = \begin{pmatrix} 51 & 23 + 8j \\ 23 - 8j & 37 \end{pmatrix}, \tag{80}$$

and

$$\mathbf{D} = \begin{pmatrix} 32 & 17 + 9j \\ 17 - 9j & 73 \end{pmatrix}. \tag{81}$$

We have determined that $\mathbf{N}$ and $\mathbf{D}$ are positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. For all nonzero $\mathbf{x} \in \mathbb{C}^2$, r has a maximum value $r_{\max}$, which may be computed as the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ to obtain $r_{\max} \simeq 1.618176$, and r has a minimum value $r_{\min}$, which may be computed as the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ to obtain $r_{\min} \simeq 0.406748$. We find

$$\frac{\operatorname{tr}\,(\mathbf{N}\mathbf{D}^{-1})}{\nu} \simeq 1.012462 \tag{82}$$

and

$$\frac{\operatorname{tr} \mathbf{N}}{\operatorname{tr} \mathbf{D}} \simeq 0.838095\,. \tag{83}$$

Fig. 1 shows the denominator probability density f_{T}, computed using (52). For $\chi_1 = 0$ and any values of ϕ_0 and ϕ_1, we have $f_{\mathrm{T}} \simeq 0.070442$. For $\chi_1 = \pi/2$ and any values of ϕ_0 and ϕ_1, we have $f_{\mathrm{T}} \simeq 0.030879$. Using a numerical integration, (20) and (52), we get

$$\underset{\hat{\mathbf{x}} \in \mathbb{S}_\nu}{\iiint} f_{\mathrm{T}}(\hat{\mathbf{x}})\, dS_\nu \simeq 1.000000\,, \tag{84}$$

which agrees with (30). Using (20), (52) and a numerical integration in (31), we get

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{T}}} \simeq 0.838095\,. \tag{85}$$

We observe that (83) and (85) are compatible with (55).

Fig. 2 shows the surface-element-ratio probability density f_{W}, computed using (13), (60), (70), (72) and (96). Note that $\hat{\mathbf{x}} \in \Omega$ if and only if $\chi_1 = 0$ or $\chi_1 = \pi/2$. Using a numerical integration, (60), (70), (73) and (96) we get

$$\underset{\hat{\mathbf{x}} \in \mathbb{S}_\nu}{\iiint} f_{\mathrm{W}}(\hat{\mathbf{x}})\, dS_\nu \simeq 1.000000\,, \tag{86}$$

which agrees with (30). Using (60), (70), (73), (96) and a numerical integration in (31), we get

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} \simeq 1.012462\,. \tag{87}$$

We observe that (82) and (87) are compatible with (78).

VIII. SECOND EXAMPLE

In a second example, $\nu = 3$,

$$\mathbf{N} = \begin{pmatrix} 41 & -11 + 2j & 5 - 13j \\ -11 - 2j & 15 & 2 - 3j \\ 5 + 13j & 2 + 3j & 22 \end{pmatrix}, \tag{88}$$

and

$$\mathbf{D} = \begin{pmatrix} 11 & 4 - 8j & -2 + 2j \\ 4 + 8j & 22 & 7 + 3j \\ -2 - 2j & 7 - 3j & 17 \end{pmatrix}. \tag{89}$$

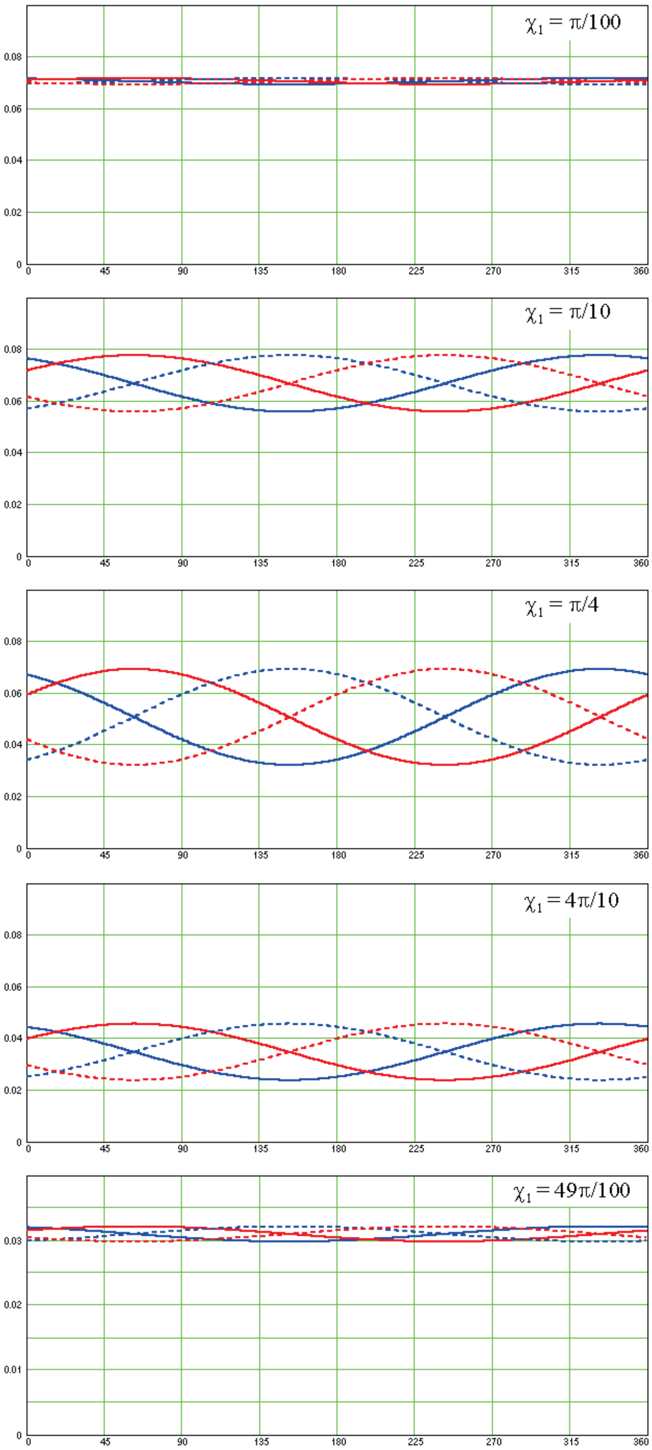

FIGURE 1. f_{T} of Section VII, for five values of χ_1, as a function of ϕ_0 in degrees: blue solid curve for $\phi_1 = 0$; red solid curve for $\phi_1 = \pi/2$; blue dashed curve for $\phi_1 = \pi$; and red dashed curve for $\phi_1 = 3\pi/2$.

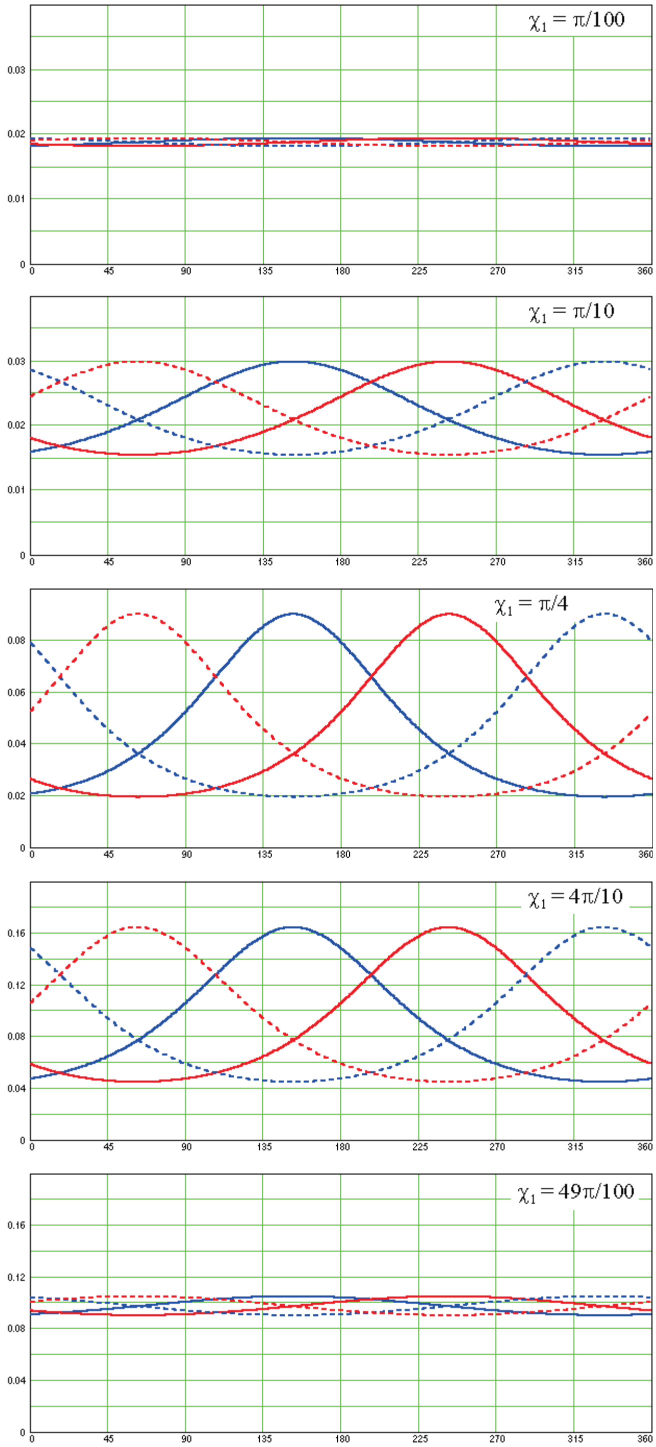

FIGURE 2. f_{W} of Section VII, for five values of χ_1, as a function of ϕ_0 in degrees: blue solid curve for $\phi_1 = 0$; red solid curve for $\phi_1 = \pi/2$; blue dashed curve for $\phi_1 = \pi$; and red dashed curve for $\phi_1 = 3\pi/2$.

We have determined that $\mathbf{N}$ and $\mathbf{D}$ are positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. For all nonzero $\mathbf{x} \in \mathbb{C}^3$, r has a maximum value $r_{\max}$, which may be computed as the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ to obtain $r_{\max} \simeq 14.436577$, and r has a minimum value $r_{\min}$, which may be computed as the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ to obtain $r_{\min} \simeq 0.246923$.

We find

$$\frac{\operatorname{tr}(\mathbf{N}\mathbf{D}^{-1})}{\nu} \simeq 5.267995 \tag{90}$$

and

$$\frac{\operatorname{tr}\mathbf{N}}{\operatorname{tr}\mathbf{D}} \simeq 1.560000\,. \tag{91}$$

Using a numerical integration, (20) and (52), we get

$$\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} \!\!\cdots\!\! \int f_{\mathrm{T}}(\hat{\mathbf{x}})\, dS_\nu \simeq 1.000000\,, \tag{92}$$

which agrees with (30). Using (20), (52) and a numerical integration in (31), we get

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{T}}} \simeq 1.560000\,. \tag{93}$$

We observe that (91) and (93) are compatible with (55).

Using a numerical integration, (60), (70), (73) and (100)–(104), we get

$$\int_{\hat{\mathbf{x}} \in \mathbb{S}_\nu} \!\!\cdots\!\! \int f_{\mathrm{W}}(\hat{\mathbf{x}})\, dS_\nu \simeq 1.000000\,, \tag{94}$$

which agrees with (30). Using (60), (70), (73), (100)–(104) and a numerical integration in (31), we get

$$\langle r(\hat{\mathbf{x}}) \rangle_{f_{\mathrm{W}}} \simeq 5.267995\,. \tag{95}$$

We observe that (90) and (95) are compatible with (78).

To compute the same quantities for the problems of Section VII and Section VIII, the run time of our program was about 6980 times longer for Section VIII than for Section VII. This confirms that the computation time of the multiple integrals evaluated in this program increases rapidly when ν increases. For this reason, we did not try to perform the same computations for any problem in which $\nu \geqslant 4$.

IX. CONCLUSION

Let ν be a positive integer and let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, where $\mathbf{D}$ is positive definite. For a generalized Rayleigh ratio r of $\mathbf{N}$ to $\mathbf{D}$ and a random nonzero excitation $\mathbf{x} \in \mathbb{C}^\nu$, we have defined a probability density of the normalized random excitation, and considered the corresponding expected value of r.

If the probability density of the normalized random excitation is a function, the definition of the expected value of r involves a multiple integral of a function of $2\nu - 1$ variables over the hypersphere of $\mathbb{R}^{2\nu}$. The computational cost of an accurate numerical evaluation of this multiple integral is typically: a fast growing function of ν; and assuredly too high for the authors if $\nu \geqslant 5$.

We have identified three remarkable probability densities of the normalized random excitation, denoted by f_{D}, f_{T} and f_{W}, respectively, for which a computation of a multiple integral is not needed to determine the expected value of r. The uniform probability density f_{U} is not one of them, but we have $f_{\mathrm{U}} = f_{\mathrm{T}} = f_{\mathrm{W}}$ in the special case $\mathbf{D} = d\,\mathbf{1}_\nu$, where d is a positive real.

A preliminary result of this article is that f_{T}, referred to as the "denominator probability density", only depends on $\mathbf{D}$ and is such that the expected value of r is the ratio of the trace of $\mathbf{N}$ to the trace of $\mathbf{D}$. The computation of f_{T} is straightforward.

The main result of this article is that f_{W}, referred to as the "surface-element-ratio probability density", only depends on $\mathbf{D}$ and is such that the expected value of r is the ratio of the trace of $\mathbf{N}\mathbf{D}^{-1}$ to ν. We have thoroughly explained the computation of f_{W}.

APPENDIX

A. INTRODUCTORY OBSERVATION

In this Appendix, we compute the jacobian matrix $\mathbf{J}$ given by (11) and the metric matrix $\mathbf{G}_{\mathrm{M}}$ given by (13), which were used to obtain dS_ν in Section IV. The jacobian matrix $\mathbf{J}$ is needed to compute the jacobian matrix $\mathbf{J}'$ and then f_{W} in Section VI.

In what follows, we also discuss Ω, defined as the set of the $\hat{\mathbf{x}} \in \mathbb{S}_\nu$ such that $\det \mathbf{G}_{\mathrm{M}} = 0$, or equivalently $dS_\nu = 0$.

B. CASE $\nu = 2$

For $\nu = 2$, it follows from (19) that

$$\mathbf{J} = \begin{pmatrix} \cos\chi_1 \cos\phi_1 & 0 & -\sin\chi_1 \sin\phi_1 \\ -\sin\chi_1 \cos\phi_0 & -\cos\chi_1 \sin\phi_0 & 0 \\ \cos\chi_1 \sin\phi_1 & 0 & \sin\chi_1 \cos\phi_1 \\ -\sin\chi_1 \sin\phi_0 & \cos\chi_1 \cos\phi_0 & 0 \end{pmatrix}, \tag{96}$$

so that

$$\mathbf{G}_{\mathrm{M}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\chi_1 & 0 \\ 0 & 0 & \sin^2\chi_1 \end{pmatrix}. \tag{97}$$

Thus, for $\nu = 2$, we obtain

$$\sqrt{\det \mathbf{G}_{\mathrm{M}}} = \cos\chi_1 \sin\chi_1 \tag{98}$$

and

$$dS_2 = \cos\chi_1 \sin\chi_1 \; d\zeta_1\, d\zeta_2\, d\zeta_3\,. \tag{99}$$

It follows from (99) that $\hat{\mathbf{x}} \in \Omega$ if and only if $\chi_1 = 0$ or $\chi_1 = \pi/2$. Thus, by (19), Ω is the union of two circles contained in $\mathbb{S}_2$.

C. CASE $\nu = 3$

For $\nu = 3$, it follows from (23) that

$$\mathbf{J}^{\langle 1 \rangle} = \begin{pmatrix} \cos\chi_1 \sin\chi_2 \cos\phi_2 \\ \cos\chi_1 \cos\chi_2 \cos\phi_1 \\ -\sin\chi_1 \cos\phi_0 \\ \cos\chi_1 \sin\chi_2 \sin\phi_2 \\ \cos\chi_1 \cos\chi_2 \sin\phi_1 \\ -\sin\chi_1 \sin\phi_0 \end{pmatrix}, \tag{100}$$

$$\mathbf{J}^{\langle 2 \rangle} = \begin{pmatrix} \sin\chi_1 \cos\chi_2 \cos\phi_2 \\ -\sin\chi_1 \sin\chi_2 \cos\phi_1 \\ 0 \\ \sin\chi_1 \cos\chi_2 \sin\phi_2 \\ -\sin\chi_1 \sin\chi_2 \sin\phi_1 \\ 0 \end{pmatrix}, \tag{101}$$

$$\mathbf{J}^{\langle 3 \rangle} = \begin{pmatrix} 0 \\ 0 \\ -\cos\chi_1 \sin\phi_0 \\ 0 \\ 0 \\ \cos\chi_1 \cos\phi_0 \end{pmatrix}, \tag{102}$$

$$\mathbf{J}^{\langle 4\rangle} = \begin{pmatrix} 0 \\ -\sin\chi_1\cos\chi_2\sin\phi_1 \\ 0 \\ 0 \\ \sin\chi_1\cos\chi_2\cos\phi_1 \\ 0 \end{pmatrix}, \tag{103}$$

and

$$\mathbf{J}^{\langle 5\rangle} = \begin{pmatrix} -\sin\chi_1\sin\chi_2\sin\phi_2 \\ 0 \\ 0 \\ \sin\chi_1\sin\chi_2\cos\phi_2 \\ 0 \\ 0 \end{pmatrix}, \tag{104}$$

so that

$$\mathbf{G}_\mathrm{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sin^2\chi_1 & 0 & 0 & 0 \\ 0 & 0 & \cos^2\chi_1 & 0 & 0 \\ 0 & 0 & 0 & \sin^2\chi_1\cos^2\chi_2 & 0 \\ 0 & 0 & 0 & 0 & \sin^2\chi_1\sin^2\chi_2 \end{pmatrix}. \tag{105}$$

Thus, for $\nu = 3$, we obtain

$$\sqrt{\det \mathbf{G}_\mathrm{M}} = \cos\chi_1\sin^3\chi_1\cos\chi_2\sin\chi_2 \tag{106}$$

and

$$dS_3 = \cos\chi_1\sin^3\chi_1\cos\chi_2\sin\chi_2 \; d\zeta_1\cdots d\zeta_5\,. \tag{107}$$

It follows from (107) that $\hat{\mathbf{x}} \in \Omega$ if and only if $\chi_1 = 0$ or $\chi_1 = \pi/2$ or $\chi_2 = 0$ or $\chi_2 = \pi/2$. Thus, by (23), Ω is the union of four subsets of $\mathbb{S}_3$: a circle and 3 differentiable manifolds of dimension 3.

D. CASE $\nu \geqslant 4$

If m and n lie in $\mathbb{N}$ and $m > n$, we posit $\prod_{i=m}^{n} x_i = 1$ no matter what the quantities $x_n, \cdots, x_m$ are.

For $\nu \geqslant 4$, it follows from (27) that

$$\mathbf{J}^{\langle 1\rangle} = \begin{pmatrix} \cos\chi_1\left(\prod_{i=2}^{\nu-1}\sin\chi_i\right)\cos\phi_{\nu-1} \\ \cos\chi_1\left(\prod_{i=2}^{\nu-2}\sin\chi_i\right)\cos\chi_{\nu-1}\cos\phi_{\nu-2} \\ \vdots \\ \cos\chi_1\cos\chi_2\cos\phi_1 \\ -\sin\chi_1\cos\phi_0 \\ \cos\chi_1\left(\prod_{i=2}^{\nu-1}\sin\chi_i\right)\sin\phi_{\nu-1} \\ \cos\chi_1\left(\prod_{i=2}^{\nu-2}\sin\chi_i\right)\cos\chi_{\nu-1}\sin\phi_{\nu-2} \\ \vdots \\ \cos\chi_1\cos\chi_2\sin\phi_1 \\ -\sin\chi_1\sin\phi_0 \end{pmatrix}, \tag{108}$$

and

$$\mathbf{J}^{\langle 2\rangle} = \begin{pmatrix} \cos\chi_2\left(\prod_{\substack{i=1\\ i\neq 2}}^{\nu-1}\sin\chi_i\right)\cos\phi_{\nu-1} \\ \cos\chi_2\left(\prod_{\substack{i=1\\ i\neq 2}}^{\nu-2}\sin\chi_i\right)\cos\chi_{\nu-1}\cos\phi_{\nu-2} \\ \vdots \\ \cos\chi_2\sin\chi_1\cos\chi_3\cos\phi_2 \\ -\sin\chi_2\sin\chi_1\cos\phi_1 \\ 0 \\ \cos\chi_2\left(\prod_{\substack{i=1\\ i\neq 2}}^{\nu-1}\sin\chi_i\right)\sin\phi_{\nu-1} \\ \cos\chi_2\left(\prod_{\substack{i=1\\ i\neq 2}}^{\nu-2}\sin\chi_i\right)\cos\chi_{\nu-1}\sin\phi_{\nu-2} \\ \vdots \\ \cos\chi_2\sin\chi_1\cos\chi_3\sin\phi_2 \\ -\sin\chi_2\sin\chi_1\sin\phi_1 \\ 0 \end{pmatrix}. \tag{109}$$

If $\nu \geqslant 6$ and $p \in \{3, \ldots, \nu-3\}$, we find that $\mathbf{J}^{\langle p\rangle}$ is given by

$$\mathbf{J}^{\langle p\rangle} = \begin{pmatrix} \cos\chi_p\left(\prod_{\substack{i=1\\ i\neq p}}^{\nu-1}\sin\chi_i\right)\cos\phi_{\nu-1} \\ \cos\chi_p\left(\prod_{\substack{i=1\\ i\neq p}}^{\nu-2}\sin\chi_i\right)\cos\chi_{\nu-1}\cos\phi_{\nu-2} \\ \vdots \\ \cos\chi_p\left(\prod_{i=1}^{p-1}\sin\chi_i\right)\cos\chi_{p+1}\cos\phi_p \\ -\sin\chi_p\left(\prod_{i=1}^{p-1}\sin\chi_i\right)\cos\phi_{p-1} \\ 0 \\ \vdots \\ 0 \\ \cos\chi_p\left(\prod_{\substack{i=1\\ i\neq p}}^{\nu-1}\sin\chi_i\right)\sin\phi_{\nu-1} \\ \cos\chi_p\left(\prod_{\substack{i=1\\ i\neq p}}^{\nu-2}\sin\chi_i\right)\cos\chi_{\nu-1}\sin\phi_{\nu-2} \\ \vdots \\ \cos\chi_p\left(\prod_{i=1}^{p-1}\sin\chi_i\right)\cos\chi_{p+1}\sin\phi_p \\ -\sin\chi_p\left(\prod_{i=1}^{p-1}\sin\chi_i\right)\sin\phi_{p-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{110}$$

where the entries of the rows $\nu-p+2$ to ν and $2\nu-p+2$ to 2ν are zero.

If $\nu \geqslant 5$, we obtain:

$$\mathbf{J}^{\langle \nu-2 \rangle} = \begin{pmatrix} \sin\chi_1 \cdots \sin\chi_{\nu-3} \cos\chi_{\nu-2} \sin\chi_{\nu-1} \cos\phi_{\nu-1} \\ \sin\chi_1 \cdots \sin\chi_{\nu-3} \cos\chi_{\nu-2} \cos\chi_{\nu-1} \cos\phi_{\nu-2} \\ -\sin\chi_1 \cdots \sin\chi_{\nu-3} \sin\chi_{\nu-2} \cos\phi_{\nu-3} \\ 0 \\ \vdots \\ 0 \\ \sin\chi_1 \cdots \sin\chi_{\nu-3} \cos\chi_{\nu-2} \sin\chi_{\nu-1} \sin\phi_{\nu-1} \\ \sin\chi_1 \cdots \sin\chi_{\nu-3} \cos\chi_{\nu-2} \cos\chi_{\nu-1} \sin\phi_{\nu-2} \\ -\sin\chi_1 \cdots \sin\chi_{\nu-3} \sin\chi_{\nu-2} \sin\phi_{\nu-3} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{111}$$

where the entries of the rows 4 to ν and $\nu+4$ to 2ν are zero. Going back to the case $\nu \geqslant 4$, we obtain:

$$\mathbf{J}^{\langle \nu-1 \rangle} = \begin{pmatrix} \sin\chi_1 \cdots \sin\chi_{\nu-2} \cos\chi_{\nu-1} \cos\phi_{\nu-1} \\ -\sin\chi_1 \cdots \sin\chi_{\nu-2} \sin\chi_{\nu-1} \cos\phi_{\nu-2} \\ 0 \\ \vdots \\ 0 \\ \sin\chi_1 \cdots \sin\chi_{\nu-2} \cos\chi_{\nu-1} \sin\phi_{\nu-1} \\ -\sin\chi_1 \cdots \sin\chi_{\nu-2} \sin\chi_{\nu-1} \sin\phi_{\nu-2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{112}$$

where the entries of the rows 3 to ν and $\nu+3$ to 2ν are zero;

$$\mathbf{J}^{\langle \nu \rangle} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\cos\chi_1 \sin\phi_0 \\ 0 \\ \vdots \\ 0 \\ \cos\chi_1 \cos\phi_0 \end{pmatrix}, \tag{113}$$

where the entries of the rows 1 to $\nu-1$ and $\nu+1$ to $2\nu-1$ are zero; and

$$\mathbf{J}^{\langle \nu+1 \rangle} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\sin\chi_1 \cos\chi_2 \sin\phi_1 \\ 0 \\ \vdots \\ 0 \\ \sin\chi_1 \cos\chi_2 \cos\phi_1 \\ 0 \end{pmatrix}, \tag{114}$$

where the entries of the rows 1 to $\nu-2$, ν to $2\nu-2$, and 2ν are zero.

If $\nu \geqslant 5$ and $q \in \{\nu+2, \dots, 2\nu-3\}$ we find that $\mathbf{J}^{\langle q \rangle}$ is given by

$$\mathbf{J}^{\langle q \rangle} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\sin\chi_1 \cdots \sin\chi_{q-\nu} \cos\chi_{q-\nu+1} \sin\phi_{q-\nu} \\ 0 \\ \vdots \\ 0 \\ \sin\chi_1 \cdots \sin\chi_{q-\nu} \cos\chi_{q-\nu+1} \cos\phi_{q-\nu} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{115}$$

where the entries of the rows 1 to $2\nu-q-1$, $2\nu-q+1$ to $3\nu-q-1$ and $3\nu-q+1$ to 2ν are zero. Going back to the case $\nu \geqslant 4$, we obtain:

$$\mathbf{J}^{\langle 2\nu-2 \rangle} = \begin{pmatrix} 0 \\ -\sin\chi_1 \cdots \sin\chi_{\nu-2} \cos\chi_{\nu-1} \sin\phi_{\nu-2} \\ 0 \\ \vdots \\ 0 \\ \sin\chi_1 \cdots \sin\chi_{\nu-2} \cos\chi_{\nu-1} \cos\phi_{\nu-2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{116}$$

where the entries of the rows 1, 3 to $\nu+1$ and $\nu+3$ to 2ν are zero; and

$$\mathbf{J}^{\langle 2\nu-1 \rangle} = \begin{pmatrix} -\sin\chi_1 \cdots \sin\chi_{\nu-1} \sin\phi_{\nu-1} \\ 0 \\ \vdots \\ 0 \\ \sin\chi_1 \cdots \sin\chi_{\nu-1} \cos\phi_{\nu-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{117}$$

where the entries of the rows 2 to ν and $\nu+2$ to 2ν are zero.

For any p and q lying in $\{1, \dots, 2\nu-1\}$, a rather tedious computation leads to:

$$(p \neq q) \Longrightarrow (\mathbf{J}^{\langle p \rangle \mathrm{T}} \, \mathbf{J}^{\langle q \rangle} = 0). \tag{118}$$

Consequently, $\mathbf{G}_{\mathrm{M}}$ is a diagonal matrix. Concerning the diagonal entries of $\mathbf{G}_{\mathrm{M}}$, we find:

$$\mathbf{J}^{\langle 1 \rangle \mathrm{T}} \, \mathbf{J}^{\langle 1 \rangle} = 1\,; \tag{119}$$

$$\mathbf{J}^{\langle 2 \rangle \mathrm{T}} \, \mathbf{J}^{\langle 2 \rangle} = \sin^2\chi_1\,; \tag{120}$$

for any p lying in $\{3, \dots, \nu-1\}$,

$$\mathbf{J}^{\langle p \rangle \mathrm{T}} \, \mathbf{J}^{\langle p \rangle} = \sin^2\chi_1 \cdots \sin^2\chi_{p-1}\,; \tag{121}$$

$$\mathbf{J}^{\langle \nu \rangle \mathrm{T}} \, \mathbf{J}^{\langle \nu \rangle} = \cos^2 \chi_1 \, ; \tag{122}$$

$$\mathbf{J}^{\langle \nu+1 \rangle \mathrm{T}} \, \mathbf{J}^{\langle \nu+1 \rangle} = \sin^2 \chi_1 \cos^2 \chi_2 \, ; \tag{123}$$

for any q lying in $\{\nu + 2, \ldots, 2\nu - 2\}$,

$$\mathbf{J}^{\langle q \rangle \mathrm{T}} \, \mathbf{J}^{\langle q \rangle} = \sin^2 \chi_1 \cdots \sin^2 \chi_{q-\nu} \cos^2 \chi_{q-\nu+1} \, ; \tag{124}$$

and

$$\mathbf{J}^{\langle 2\nu-1 \rangle \mathrm{T}} \, \mathbf{J}^{\langle 2\nu-1 \rangle} = \sin^2 \chi_1 \cdots \sin^2 \chi_{\nu-1} \, . \tag{125}$$

Thus, for $\nu \geqslant 4$, we obtain

$$\sqrt{\det \mathbf{G}_{\mathrm{M}}} = \cos \chi_1 \sin^{2\nu-3} \chi_1 \times \cos \chi_2 \sin^{2\nu-5} \chi_2 \cdots \cos \chi_{\nu-1} \sin \chi_{\nu-1} \tag{126}$$

and

$$dS_\nu = \cos \chi_1 \sin^{2\nu-3} \chi_1 \cos \chi_2 \sin^{2\nu-5} \chi_2 \cdots \times \cos \chi_{\nu-1} \sin \chi_{\nu-1} \; d\zeta_1 \cdots d\zeta_{2\nu-1} \, . \tag{127}$$

It follows from (127) that, for $\nu \geqslant 4$, $\hat{\mathbf{x}} \in \Omega$ if and only if $\chi_1 = 0$ or $\chi_1 = \pi/2$ or ... or $\chi_{\nu-1} = 0$ or $\chi_{\nu-1} = \pi/2$. Thus, by (27), Ω is the union of $2\nu-2$ subsets of $\mathbb{S}_\nu$. If $\nu = 4$, these $2\nu - 2 = 6$ subsets are: one circle, one differentiable manifold of dimension 3, and 4 differentiable manifolds of dimension 5. If $\nu \geqslant 5$, these $2\nu-2$ subsets are: one circle, one differentiable manifold of dimension 3, ... , one differentiable manifold of dimension $2\nu - 5$, and ν differentiable manifolds of dimension $2\nu - 3$.

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FRÉDÉRIC BROYDÉ was born in France in 1960. He received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble (ENSIEG) and the Ph.D. in microwaves and microtechnologies from the Université des Sciences et Technologies de Lille (USTL).

He co-founded the Excem corporation in May 1988, a company providing engineering and research and development services. He is president of Excem since 1988. He is now also president and CTO of Eurexcem, a subsidiary of Excem. Most of his activity is allocated to engineering and research in electronics, radio, antennas, electromagnetic compatibility (EMC) and signal integrity.

Dr. Broydé is author or co-author of about 100 technical papers, and inventor or co-inventor of about 90 patent families, for which 72 patents of France and 49 patents of the USA have been granted. He became a Senior Member of the IEEE in 2001. He is a licensed radio amateur (F5OYE).

EVELYNE CLAVELIER was born in France in 1961. She received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble (ENSIEG).

She is co-founder of the Excem corporation, based in Maule, France, and she is currently CEO of Excem. She is also president of Tekcem, a company selling or licensing intellectual property rights to foster research. She is an active engineer and researcher.

Her current research areas are radio communications, antennas, matching networks, EMC and circuit theory.

Prior to starting Excem in 1988, she worked for Schneider Electrics (in Grenoble, France), STMicroelectronics (in Grenoble, France), and Signetics (in Mountain View, CA, USA).

Ms. Clavelier is the author or a co-author of about 90 technical papers. She is co-inventor of about 90 patent families. She is a Senior Member of the IEEE since 2002. She is a licensed radio amateur (F1PHQ).

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