Excem Research Papers in Electronics and Electromagnetics

Number 10 January 2025 ISBN 978-2-909056-30-2

About the Gains and the Effective Areas of a Multiport Antenna Array

FRÉDÉRIC BROYDÉ[1], and
EVELYNE CLAVELIER[2]

[1]Eurexcem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France
[2]Excem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France

Corresponding author: Frédéric Broydé (e-mail: fredbroyde@eurexcem.com).

ABSTRACT We study emission by a multiport antenna array (MAA) coupled to a multiport generator having an internal impedance matrix that need not be diagonal, and reception by the MAA coupled to a multiport load having an impedance matrix that need not be diagonal. This leads us to define 4 excitation-dependent emission parameters (gains), 10 excitation-independent emission parameters, and 8 reception parameters including two effective areas. We establish how these parameters can be computed and prove many results about them. We use the parameters for emission and reception to obtain new generalizations of the Friis transmission formula and other new transmission formulas.

INDEX TERMS Antenna array, partial absolute gain, absolute gain, partial reached gain, reached gain, partial absolute effective area, absolute effective area, partial reached effective area, reached effective area, MIMO, reciprocity, antenna theory, Friis transmission formula.

I. INTRODUCTION

The radiation intensity produced in a given direction by a single-port antenna emitting at a given frequency is completely determined by the absolute gain of the antenna in this direction and the average power received by the port of the antenna [1]–[2]. Moreover, if this average power is not known but the available power of a generator coupled to the antenna is specified, the radiation intensity can be computed using the absolute gain and the power transfer ratio between the generator and the antenna [3].

The radiation intensity produced by a multiport antenna array (MAA) used for emission is much more involved, because, if the average power received by the ports of the MAA is fixed, the radiation intensity and the polarization of the radiated field also depend on the excitation applied to the MAA, more precisely on the relative phases and amplitudes of the signals at the ports of the MAA. Moreover, the power transfer ratio between a multiport generator and the MAA also depends on the excitation [4].

As regards reception at the given frequency, the available power of a single-port antenna receiving a uniform plane wave from a given direction depends on the polarization of the plane wave in such a way that it has a minimum equal to zero, and a maximum equal to the effective area of the antenna in this direction times the power flux density of the plane wave [1]–[2]. Moreover, the average power delivered to a load coupled to the antenna has a minimum equal to zero and a maximum equal to the product of the maximum available power and the power transfer ratio between the antenna and the load.

The available power of an MAA used for reception is more complicated, because the dependence of the available power on the polarization of the incident plane wave need not be similar to the one applicable to a single-port antenna. Moreover, the power transfer ratio between the MAA and a multiport load also depends on the polarization, so that the polarization that maximizes the available power need not correspond to the polarization that maximizes the average power delivered to the multiport load.

The stated difficulties involving an MAA are not addressed in the current IEC and IEEE standards defining the vocabulary relating to antennas [1]–[2]. To investigate these difficulties, we consider a linear time-invariant (LTI) MAA, which need not be reciprocal. When it is used for emission, the MAA is coupled to a multiport generator having an internal impedance matrix that need not be diagonal, and we investigate the radiation intensity produced by the MAA. When it is used for reception, the MAA is coupled to a multiport load having an impedance matrix that need not be diagonal, and we investigate the available power and the average power delivered by the ports of the MAA.

This work allows us to introduce suitable parameters of the MAA, which address the above-mentioned difficulties. Using configurations involving a remote antenna array

TABLE 1. Excitation-dependent parameters for emission.

Quantity	Symbol	Section
partial absolute gain	$G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$	IV
absolute gain	G_{a}	V
partial reached gain	$G_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})$	VII
reached gain	G_{r}	VIII

TABLE 2. Excitation-independent parameters for emission.

Quantity	Symbol	Section
maximum partial absolute gain	$G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$	IV
mean partial absolute gain	$G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}})$	IV
maximum absolute gain	$G_{\mathrm{a\,MAX}}$	V
mean absolute gain	$G_{\mathrm{a\,MEA}}$	VI
minimum absolute gain	$G_{\mathrm{a\,MIN}}$	V
maximum partial reached gain	$G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})$	VII
mean partial reached gain	$G_{\mathrm{pr\,MEA}}(\mathbf{u}_{\mathrm{pol}})$	VII
maximum reached gain	$G_{\mathrm{r\,MAX}}$	VIII
mean reached gain	$G_{\mathrm{r\,MEA}}$	IX
minimum reached gain	$G_{\mathrm{r\,MIN}}$	VIII

TABLE 3. Reception parameters.

Quantity	Symbol	Section
partial absolute effective area	$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$	IV
absolute effective area	A_{a}	V
mean absolute equivalent area	$A_{\mathrm{aeq\,MEA}}$	VI
minimum absolute equivalent area	$A_{\mathrm{aeq\,MIN}}$	VI
partial reached effective area	$A_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})$	VII
reached effective area	A_{r}	VIII
mean reached equivalent area	$A_{\mathrm{req\,MEA}}$	IX
minimum reached equivalent area	$A_{\mathrm{req\,MIN}}$	IX

(RAA) and results disclosed in [5] and [6], we establish important relationships among these parameters, which occur when the MAA is reciprocal and other conditions are met. As regards emission, we study the excitation-dependent parameters listed in Table 1 to obtain the wanted excitation-independent parameters listed in Table 2. As regards reception, the wanted parameters listed in Table 3 are obtained directly.

The article is organized as follows. Generalized Rayleigh ratios are briefly introduced in Section II. Our assumptions, notations, the definitions of the configurations comprising the MAA and the RAA, and simple results about them are provided in Section III. Sections IV to IX are used to define the parameters listed in Table 2 and Table 3, and to prove 8 relationships involving the reciprocity of the MAA. Sections X to XII provide additional results, a generalization and a simple example. In Section XIII and Section XIV, our results are used to obtain new transmission formulas.

II. GENERALIZED RAYLEIGH RATIO

Let ν be a positive integer. The vector space of the complex column vectors of size ν is denoted by $\mathbb{C}^\nu$.

We use $\mathbf{1}_\nu$ to denote the identity matrix of size ν by ν. Let $\mathbf{M}$ be a complex matrix. We use $\ker \mathbf{M}$ to denote the nullspace of $\mathbf{M}$, $\mathrm{rank}\,\mathbf{M}$ the rank of $\mathbf{M}$, $\mathbf{M}^T$ the transpose of $\mathbf{M}$, and $\mathbf{M}^*$ the hermitian adjoint of $\mathbf{M}$. If $\mathbf{M}$ is square, $\mathrm{tr}\,\mathbf{M}$ denotes the trace of $\mathbf{M}$ and $H(\mathbf{M})$ denotes the hermitian part of $\mathbf{M}$ given by

$$H(\mathbf{M}) = \frac{\mathbf{M}+\mathbf{M}^*}{2}\,. \tag{1}$$

Let $\mathbf{A}$ be a positive semidefinite matrix. We know [7, Sec. 7.2.6] that there exists a unique positive semidefinite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. The matrix $\mathbf{B}$ is referred to as the unique positive semidefinite square root of $\mathbf{A}$, and is denoted by $\mathbf{A}^{1/2}$. If $\mathbf{A}$ is positive definite, $\mathbf{A}^{-1}$ and $\mathbf{A}^{1/2}$ are positive definite, and $(\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$, so that we can write $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$.

Let $\mathbf{A}$ be a hermitian matrix of size ν by ν. The expression $\mathbf{x}^*\mathbf{A}\mathbf{x}/\mathbf{x}^*\mathbf{x}$, where $\mathbf{x} \in \mathbb{C}^\nu$, is known as a Rayleigh ratio, or Rayleigh-Ritz ratio, or Rayleigh quotient [7, Sec. 4.2], [8, Sec. 4.2]. In this article, this concept is extended as follows. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive semidefinite. The generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$ is a real-valued function $r : \mathbb{C}^\nu \to \mathbb{R}$ such that, for any $\mathbf{x} \in \mathbb{C}^\nu$ satisfying $\mathbf{x}^*\mathbf{D}\mathbf{x} \neq 0$, we have

$$r(\mathbf{x}) = \frac{\mathbf{x}^*\mathbf{N}\mathbf{x}}{\mathbf{x}^*\mathbf{D}\mathbf{x}}\,. \tag{2}$$

Let $\mathbf{A}$ be a positive semidefinite matrix of size ν by ν. We know [7, Sec. 7.1.6] that, for any $\mathbf{x} \in \mathbb{C}^\nu$, $\mathbf{x}^*\,\mathbf{A}\,\mathbf{x} = 0$ if and only if $\mathbf{x} \in \ker \mathbf{A}$. In this article, we further assume that $\mathbf{D}$ is positive definite, so that $r(\mathbf{x})$ is defined for any nonzero $\mathbf{x}$.

Let $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^*\mathbf{x}}$ be the euclidian vector norm of an arbitrary complex column vector $\mathbf{x}$. We use $\mathbb{S}_\nu$ to denote the hypersphere of the unit vectors of $\mathbb{C}^\nu$. It follows from (2) that, for $\mathbf{x} \neq \mathbf{0}$ and a fixed $\mathbf{x}/||\mathbf{x}||_2$, $r(\mathbf{x})$ does not depend on $||\mathbf{x}||_2$. Thus, the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \neq \mathbf{0}$ is equal to the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \in \mathbb{S}_\nu$.

Theorem [6, Sec. II], [9, Sec. II]). Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size ν by ν, $\mathbf{D}$ being positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. We define

$$\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}\,. \tag{3}$$

$\mathbf{M}$ is of size ν by ν, and hermitian. Thus, its eigenvalues are real. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{M}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{M}$. For any $\mathbf{x} \in \mathbb{C}^\nu$ satisfying $\mathbf{x} \neq \mathbf{0}$, we have

$$\lambda_{\min} = \min_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} \leqslant r(\mathbf{x}) \leqslant \lambda_{\max} = \max_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}}\,. \tag{4}$$

Moreover,

- the equality $r(\mathbf{x}) = \lambda_{\max}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\max}$;
- the equality $r(\mathbf{x}) = \lambda_{\min}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\min}$; and
- $\mathbf{M}$ and $\mathbf{N}\mathbf{D}^{-1}$ are similar, so that the eigenvalues of $\mathbf{N}\mathbf{D}^{-1}$ are real, $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ and $\lambda_{\min}$ is the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$.

III. ASSUMPTIONS, NOTATIONS, SIMPLE RESULTS

A. THE CONFIGURATIONS

The available power of an LTI device having one or more ports is defined as the greatest average power that can be drawn from the LTI device by an arbitrary LTI and passive load. The reader is reminded that the available power need not be equal to an incident power [9, Sec. V]. Noise power is ignored throughout the article.

The MAA under study is LTI and lies in an isotropic, homogeneous and lossless medium. In such a medium, reciprocity holds [10, Sec. 13.06]. The MAA has N ports numbered from 1 to N. The MAA operates in the harmonic steady state, at a radian frequency ω corresponding to a wavelength λ and a wave number k in said medium.

A time factor $e^{j\omega t}$ is assumed and suppressed throughout the article. We use $\bar{z}$ to denote the complex conjugate of a complex number z. We use $\overline{\mathbf{M}}$ to denote the complex conjugate of a complex matrix $\mathbf{M}$, so that $\mathbf{M}^* = \overline{\mathbf{M}}^{\mathrm{T}}$.

We use $\mathbb{E}$ to denote the Euclidean vector space of dimension 3 associated with physical space. We use $\mathbb{E}^{\mathrm{c}}$ to denote the complex vector space of dimension 3 containing the vectors in the form $\mathbf{x} + j\mathbf{y}$, where $\mathbf{x} \in \mathbb{E}$ and $\mathbf{y} \in \mathbb{E}$. Any basis of $\mathbb{E}$ is a basis of $\mathbb{E}^{\mathrm{c}}$. For any $\mathbf{v} \in \mathbb{E}^{\mathrm{c}}$, we use $\overline{\mathbf{v}}$ to denote the complex conjugate of $\mathbf{v}$. An orthonormal basis of $\mathbb{E}$ being implicitly or explicitly chosen, for any $\mathbf{v} \in \mathbb{E}^{\mathrm{c}}$, we use $\underset{\sim}{\mathbf{v}}$ to denote the column vector of the coordinates of $\mathbf{v}$ in this basis. We have $\overline{\mathbf{v}} \cdot \mathbf{v} = \underset{\sim}{\mathbf{v}}^* \underset{\sim}{\mathbf{v}}$, and the norm of $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{\overline{\mathbf{v}} \cdot \mathbf{v}}$. Appendix A provides detailed explanations about the conjugate $\overline{\mathbf{v}}$ of $\mathbf{v}$, and on the notation $\mathbf{x} \cdot \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ lie in $\mathbb{E}^{\mathrm{c}}$.

The MAA is passive and may be used for emission and reception at ω, but it need not be reciprocal. We assume that the MAA has an impedance matrix, denoted by $\mathbf{Z}_{\mathrm{A}}$, which is of size N by N. This is equivalent to assuming that, for any integer $p \in \{1, \ldots, N\}$, it is possible to inject an arbitrary current into port p of the MAA while the other ports of the MAA are left open-circuited, the voltage across each port of the MAA being finite. The MAA being passive, $H(\mathbf{Z}_{\mathrm{A}})$ is positive semidefinite. We further assume that $H(\mathbf{Z}_{\mathrm{A}})$ is positive definite.

In physical space, we choose a right-handed rectangular cartesian coordinate system (x, y, z) having its origin, denoted by O, close to the MAA. The associated spherical coordinates system is (r, θ, φ). We use $(\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z)$ to denote the right-handed orthonormal basis of the cartesian coordinate system, and $(\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\varphi)$ to denote the local orthonormal basis of the spherical coordinate system.

To study the properties of the MAA, we use the RAA mentioned in the introduction, which also lies in the isotropic, homogeneous and lossless medium. The RAA is LTI and has n ports numbered from 1 to n, where $n \in \{1, 2\}$. The RAA can be used for reception and emission, and the RAA is reciprocal. We assume that the RAA has an impedance matrix, denoted by $\mathbf{Z}_{\mathrm{R}}$, which is of size n by n. We assume that $H(\mathbf{Z}_{\mathrm{R}})$ is positive definite. The RAA is located near a point R of coordinates $(r_{\mathrm{R}}, \theta_{\mathrm{R}}, \varphi_{\mathrm{R}})$ in the coordinate system (r, θ, φ). We assume that the distance $\mathrm{OR} = r_{\mathrm{R}}$ is much larger than the wavelength, the largest dimension of the MAA and the largest dimension of the RAA.

In fact, we have not properly defined $\mathbf{Z}_{\mathrm{A}}$ and $\mathbf{Z}_{\mathrm{R}}$. Strictly speaking, we should have said that: there is an impedance matrix of the MAA alone in said lossless medium, denoted by $\mathbf{Z}_{\mathrm{A}}$; an impedance matrix of the MAA in said lossless medium, in the presence of the RAA, the ports of the RAA being open-circuited, this matrix being denoted by $\mathbf{Z}'_{\mathrm{A}}$; and an impedance matrix of the MAA in said lossless medium, in the presence of the RAA, the ports of the RAA being coupled to a device to be specified, this matrix being denoted by $\mathbf{Z}''_{\mathrm{A}}$. Likewise, we should have said that: there is an impedance matrix of the RAA alone in said lossless medium, denoted by $\mathbf{Z}_{\mathrm{R}}$; an impedance matrix of the RAA in said lossless medium, in the presence of the MAA, the ports of the MAA being open-circuited, this matrix being denoted by $\mathbf{Z}'_{\mathrm{R}}$; and an impedance matrix of the RAA in said lossless medium, in the presence of the MAA, the ports of the MAA being coupled to a device to be specified, this matrix being denoted by $\mathbf{Z}''_{\mathrm{R}}$.

If the MAA is used for emission, the RAA is used for measuring the electromagnetic field radiated by the MAA. This is configuration A (CA). The distance r_{R} is sufficiently large to allow us to consider that:

- the RAA and the load connected to the RAA have no significant effect on the currents flowing in the MAA, so that we have $\mathbf{Z}_{\mathrm{A}} \simeq \mathbf{Z}'_{\mathrm{A}} \simeq \mathbf{Z}''_{\mathrm{A}}$;
- $\mathbf{Z}_{\mathrm{R}} \simeq \mathbf{Z}'_{\mathrm{R}} \simeq \mathbf{Z}''_{\mathrm{R}}$; and
- the electromagnetic field incident on the RAA is almost a uniform plane wave in the vicinity of the RAA.

If the MAA is used for reception, the RAA is used to generate the electromagnetic field received by the MAA. This is configuration B (CB). The distance r_{R} is sufficiently large to allow us to consider that:

- $\mathbf{Z}_{\mathrm{A}} \simeq \mathbf{Z}'_{\mathrm{A}} \simeq \mathbf{Z}''_{\mathrm{A}}$;
- the MAA and the load connected to the MAA have no significant effect on the currents flowing in the RAA, so that we have $\mathbf{Z}_{\mathrm{R}} \simeq \mathbf{Z}'_{\mathrm{R}} \simeq \mathbf{Z}''_{\mathrm{R}}$; and
- the electromagnetic field incident on the MAA is almost a uniform plane wave in the vicinity of the MAA.

Clarifications about the assumption relating to the distance r_{R} and its consequences are provided in Appendix B.

In what follows, except in Appendix B, we will use the symbol $=$ instead of the symbol $\simeq$ for results based on one of the formulas $\mathbf{Z}_{\mathrm{A}} \simeq \mathbf{Z}'_{\mathrm{A}} \simeq \mathbf{Z}''_{\mathrm{A}}$ or $\mathbf{Z}_{\mathrm{R}} \simeq \mathbf{Z}'_{\mathrm{R}} \simeq \mathbf{Z}''_{\mathrm{R}}$, because any desired accuracy in these results can be obtained by selecting a sufficiently large value of r_{R}.

Let us consider the $(N + n)$-port device whose ports are ports 1 to N of the MAA and ports 1 to n of the RAA, in this order. This $(N + n)$-port device is LTI and passive.

B. EMISSION IN CONFIGURATION A

If the MAA is used for emission, an LTI multiport generator having N ports, called MGA, is coupled to the MAA. In a setup comprising a transmitter having N antenna ports, N feeders and a N-port antenna array, the feeders may be regarded as parts of the MGA, or as parts of the MAA. The ports of the MGA are numbered from 1 to N, and, for any integer $p \in \{1,\ldots,N\}$, port p of the MGA is connected to port p of the MAA (positive terminal to positive terminal and negative terminal to negative terminal). We assume that the MGA has an internal impedance matrix $\mathbf{Z}_\mathrm{G}$ such that $H(\mathbf{Z}_\mathrm{G})$ is positive definite, or equivalently that the MGA has an admittance matrix $\mathbf{Y}_\mathrm{G}$ such that $H(\mathbf{Y}_\mathrm{G})$ is positive definite [9, Sec. IV].

In CA, the excitation of the MAA may be defined using the column vector of the rms open-circuit voltages at ports 1 to N of the MGA, denoted by $\mathbf{V}_\mathrm{AO1}$, as variable; or using other suitable variables as explained in [9, Sec. V]. $\mathbf{V}_\mathrm{AO1}$ may take on any value lying in $\mathbb{C}^N$. It follows from our assumptions and the explanations provided in [9, Sec. V] and [9, Sec. VIII] that: the matrix $\mathbf{Z}_\mathrm{A} + \mathbf{Z}_\mathrm{G}$ is invertible; the column vector of the rms currents flowing into ports 1 to N of the MAA, denoted by $\mathbf{I}_\mathrm{A1}$, can take on any value lying in $\mathbb{C}^N$ and is given by

$$\mathbf{I}_\mathrm{A1} = (\mathbf{Z}_\mathrm{A} + \mathbf{Z}_\mathrm{G})^{-1}\,\mathbf{V}_\mathrm{AO1}\,; \tag{5}$$

the available power of the MGA in CA, denoted by P_AAVG, is given by

$$P_\mathrm{AAVG} = \mathbf{I}_\mathrm{A1}^*\;\mathbf{Z}_\mathrm{AAVGO}\,\mathbf{I}_\mathrm{A1}\,, \tag{6}$$

where the impedance matrix

$$\mathbf{Z}_\mathrm{AAVGO} = (\mathbf{Z}_\mathrm{A} + \mathbf{Z}_\mathrm{G})^*\,\frac{H(\mathbf{Z}_\mathrm{G})^{-1}}{4}\,(\mathbf{Z}_\mathrm{A} + \mathbf{Z}_\mathrm{G}) \tag{7}$$

is positive definite; and the average power received by the ports of the MAA in CA, denoted by P_ARP1, is given by

$$P_\mathrm{ARP1} = \mathbf{I}_\mathrm{A1}^*\;H(\mathbf{Z}_\mathrm{A})\,\mathbf{I}_\mathrm{A1}\,. \tag{8}$$

For any integer $p \in \{1,\ldots,N\}$, we can consider a single-port antenna called SPAO-p, obtained by using only port p of the MAA alone in said medium, the other ports of the MAA being left open-circuited. Let $\mathbf{h}_{\mathrm{A}p}$ be the vector effective length of SPAO-p in a direction (θ, φ), as defined in [11]–[12], [13, Sec. 5.2] and [14, Sec. 16.5] for emission, using the origin O as reference point. We have $\mathbf{h}_{\mathrm{A}p} \cdot \mathbf{u}_r = 0$. Let $\mathbf{E}_{\mathrm{A}p}$ be the electric field radiated by SPAO-p used for emission, in the direction (θ, φ). At a large distance r of the origin O, $\mathbf{E}_{\mathrm{A}p}$ is given by

$$\mathbf{E}_{\mathrm{A}p} = j\eta\frac{I_{\mathrm{A1}\,p}\,k\,e^{-jkr}}{4\pi r}\,\mathbf{h}_{\mathrm{A}p}\,, \tag{9}$$

where η is the intrinsic impedance in the medium, and $I_{\mathrm{A1}\,p}$ is a current flowing into the port of SPAO-p.

If we now use all ports of the MAA, the linearity of the MAA entails that the electric field radiated by the MAA used for emission in the direction (θ, φ), denoted by $\mathbf{E}_\mathrm{A}$, is given by

$$\mathbf{E}_\mathrm{A} = j\eta\frac{k\,e^{-jkr}}{4\pi r}\sum_{p=1}^{N} I_{\mathrm{A1}\,p}\,\mathbf{h}_{\mathrm{A}p}\,, \tag{10}$$

where $I_{\mathrm{A1}\,1},\ldots,I_{\mathrm{A1}\,N}$ are the rms currents flowing into ports 1 to N of the MAA, that is to say, the entries of $\mathbf{I}_\mathrm{A1}$. In the derivation of (10), we have used a superposition of SPAO-1 to SPAO-N excited by the currents $I_{\mathrm{A1}\,1}$ to $I_{\mathrm{A1}\,N}$, respectively. This is possible because $\mathbf{I}_\mathrm{A1}$ may take on any value lying in $\mathbb{C}^N$. It follows that

$$\underset{\sim}{\mathbf{E}}_\mathrm{A} = j\eta\frac{k\,e^{-jkr}}{4\pi r}\,\mathbf{h}_\mathrm{A}\mathbf{I}_\mathrm{A1}\,, \tag{11}$$

where $\mathbf{h}_\mathrm{A}$ is the matrix of size 3 by N whose column vectors are $\underset{\sim}{\mathbf{h}}_\mathrm{A1}$ to $\underset{\sim}{\mathbf{h}}_{\mathrm{A}N}$, in this order. Since each column vector of the matrix $\mathbf{h}_\mathrm{A}$ lies in a plane orthogonal to $\underset{\sim}{\mathbf{u}}_r$, we have $\operatorname{rank}\mathbf{h}_\mathrm{A} \leqslant 2$. It follows that

$$\operatorname{rank}\mathbf{h}_\mathrm{A} \leqslant \min\{N, 2\}\,. \tag{12}$$

Note that $\mathbf{E}_\mathrm{A}$ and $\mathbf{h}_\mathrm{A1},\ldots,\mathbf{h}_{\mathrm{A}N}$ are vectors of $\mathbb{E}^\mathrm{c}$ in (10). In contrast, in (11), $\underset{\sim}{\mathbf{E}}_\mathrm{A}$ is a column vector and $\mathbf{h}_\mathrm{A}$ is a 3 by N matrix, both depending on the choice of an orthonormal basis of $\mathbb{E}$. If the chosen basis is $(\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\varphi)$, then all entries of the first row of $\mathbf{h}_\mathrm{A}$ are zero.

C. RECEPTION IN CONFIGURATION B

If the MAA is used for reception, an LTI multiport load having N ports, called MLA, is coupled to the MAA. The ports of the MLA are numbered from 1 to N, and, for any integer $p \in \{1,\ldots,N\}$, port p of the MLA is connected to port p of the MAA (positive terminal to positive terminal and negative terminal to negative terminal). We assume that the MLA has an impedance matrix, equal to $\mathbf{Z}_\mathrm{G}$.

For any integer $p \in \{1,\ldots,N\}$, let $\mathbf{h}_{\mathrm{B}p}$ be the vector effective length of SPAO-p in a direction (θ, φ), as defined in [15, Sec. 2.15] for reception, using the origin O as reference point (the effective length defined in [1]–[2] is related but different). Let $\mathbf{E}_\mathrm{B}$ be the electric field of an incident uniform plane wave, and $\mathbf{E}_\mathrm{B0}$ be $\mathbf{E}_\mathrm{B}$ at the origin O. The open-circuit voltage at port p of the MAA, denoted by $V_{\mathrm{BO1}\,p}$, is given by

$$V_{\mathrm{BO1}\,p} = \mathbf{h}_{\mathrm{B}p} \cdot \mathbf{E}_\mathrm{B0}\,. \tag{13}$$

Since (13) defines the vector effective length $\mathbf{h}_{\mathrm{B}p}$ for reception and we have $\mathbf{u}_r \cdot \mathbf{E}_\mathrm{B0} = 0$ for any incident uniform plane wave, $\mathbf{h}_{\mathrm{B}p} \cdot \mathbf{u}_r$ is a free parameter, and we assume that $\mathbf{h}_{\mathrm{B}p} \cdot \mathbf{u}_r = 0$.

Let $\mathbf{V}_\mathrm{BO1}$ be the column vector of the rms open-circuit voltages at ports 1 to N of the MAA during CB. It follows from (13) that

$$\mathbf{V}_\mathrm{BO1} = \mathbf{h}_\mathrm{B}^\mathrm{T}\,\underset{\sim}{\mathbf{E}}_\mathrm{B0}\,, \tag{14}$$

where $\mathbf{h}_\mathrm{B}$ is the matrix of size 3 by N whose column vectors are $\underset{\sim}{\mathbf{h}}_\mathrm{B1}$ to $\underset{\sim}{\mathbf{h}}_{\mathrm{B}N}$, in this order. Since each column vector of the matrix $\mathbf{h}_\mathrm{B}$ lies in a plane orthogonal to $\underset{\sim}{\mathbf{u}}_r$, we have $\operatorname{rank}\mathbf{h}_\mathrm{B} \leqslant 2$. It follows that

$$\operatorname{rank}\mathbf{h}_\mathrm{B} \leqslant \min\{N, 2\}\,. \tag{15}$$

Note that $\mathbf{E}_\mathrm{B0}$ and $\mathbf{h}_\mathrm{B1},\ldots,\mathbf{h}_{\mathrm{B}N}$ are vectors of $\mathbb{E}^\mathrm{c}$ in (13). In contrast, in (14), $\underset{\sim}{\mathbf{E}}_\mathrm{B0}$ is a column vector and $\mathbf{h}_\mathrm{B}$ is a 3 by N matrix, both depending on a choice of an orthonormal basis of $\mathbb{E}$. If the chosen basis is $(\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\varphi)$, then all entries of the first row of $\mathbf{h}_\mathrm{B}$ are zero.

It follows from our assumptions that the column vector of the rms currents flowing into ports 1 to N of the MAA used for reception, denoted by $\mathbf{I}_\mathrm{B1}$, is given by

$$\mathbf{I}_\mathrm{B1} = -(\mathbf{Z}_\mathrm{A} + \mathbf{Z}_\mathrm{G})^{-1}\,\mathbf{h}_\mathrm{B}^\mathrm{T}\,\underset{\sim}{\mathbf{E}}_\mathrm{B0}\,; \tag{16}$$

and the average power delivered by the ports of the MAA in CB, denoted by P_{BDP1}, is given by

$$P_{\mathrm{BDP1}} = \mathbf{I}_{\mathrm{B1}}^{*} \, H(\mathbf{Z}_{\mathrm{G}}) \, \mathbf{I}_{\mathrm{B1}} \,. \tag{17}$$

It follows from our assumptions and the explanations provided in [6, Sec. VI] that the available power of the MAA in CB, denoted by P_{BAVA}, can be computed. The general computation is involved, but we can use a simpler result because we have assumed that $H(\mathbf{Z}_{\mathrm{A}})$ is invertible:

$$P_{\mathrm{BAVA}} = \mathbf{I}_{\mathrm{B1}}^{*} \, \mathbf{Z}_{\mathrm{BAVAO}} \, \mathbf{I}_{\mathrm{B1}} \,, \tag{18}$$

where the impedance matrix

$$\mathbf{Z}_{\mathrm{BAVAO}} = (\mathbf{Z}_{\mathrm{A}} + \mathbf{Z}_{\mathrm{G}})^{*} \, \frac{H(\mathbf{Z}_{\mathrm{A}})^{-1}}{4} (\mathbf{Z}_{\mathrm{A}} + \mathbf{Z}_{\mathrm{G}}) \tag{19}$$

is positive definite.

In a situation where the MAA is reciprocal, the vector effective lengths for reception and emission coincide [11]–[12], [13, Sec. 5.2]. Thus, in such a situation, in the direction (θ, φ), we have: $\mathbf{h}_{\mathrm{A}p} = \mathbf{h}_{\mathrm{B}p}$ for any integer $p \in \{1, \dots, N\}$; and $\mathbf{h}_{\mathrm{A}} = \mathbf{h}_{\mathrm{B}}$ using any orthonormal basis of $\mathbb{E}$.

During reception in CB, an electromagnetic field produced by the RAA excites the MAA. This electromagnetic field is regarded as a uniform plane wave, of electric field $\mathbf{E}_{\mathrm{B}}$, propagating from point R, that is to say, from the direction $\theta = \theta_{\mathrm{R}}$ and $\varphi = \varphi_{\mathrm{R}}$.

D. THE REMOTE ANTENNA ARRAY

We use $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$ to denote the local orthonormal basis $(\mathbf{u}_{r}, \mathbf{u}_{\theta}, \mathbf{u}_{\varphi})$ at point R. A right-handed rectangular cartesian coordinate system (x', y', z') is such that: its origin is point R and its basis is $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$. The associated spherical coordinates system is (r', θ', φ'). In this spherical coordinate system, the coordinates of the origin O of the coordinate system (x, y, z) are $(r_{\mathrm{R}}, \pi/2, \pi)$.

For any integer $q \in \{1, \dots, n\}$, we can consider a single-port antenna called SPBO-q, obtained by using only port q of the RAA alone in said medium, the other ports of the RAA being left open-circuited. The RAA being reciprocal, we can define $\mathbf{h}_{\mathrm{R}q}$ as the vector effective length of SPBO-q in a direction (θ', φ'), for emission and reception, using R as reference point. In the direction $\theta' = \pi/2$ and $\varphi' = \pi$, we have $\mathbf{h}_{\mathrm{R}q} \cdot \mathbf{u}_{r\mathrm{R}} = 0$. An orthonormal basis of $\mathbb{E}$ being chosen, we use $\mathbf{h}_{\mathrm{R}}$ to denote the matrix of size 3 by n whose column vectors are $\underset{\sim}{\mathbf{h}}_{\mathrm{R1}}$ to $\underset{\sim}{\mathbf{h}}_{\mathrm{R}n}$, in this order.

If $n = 1$, $\mathbf{Z}_{\mathrm{R}}$ is a scalar, denoted by Z_{R} and having a positive real part (because we previously assumed that $\mathbf{Z}_{\mathrm{R}}$ is positive definite), and we further assume that, in the direction $\theta' = \pi/2$ and $\varphi' = \pi$, the antenna constituting the RAA is such that the column vector $\mathbf{h}_{\mathrm{R}} = \underset{\sim}{\mathbf{h}}_{\mathrm{R1}}$ is nonzero. Thus, we have $\operatorname{rank} \mathbf{h}_{\mathrm{R}} = 1$. For instance, this antenna may be an electrically short center-fed thin cylindrical dipole antenna, positioned in such a way that the center of the antenna is R, and that, in the direction $\theta' = \pi/2$ and $\varphi' = \pi$, we have $\mathbf{h}_{\mathrm{R1}} = h_{\mathrm{R}} \, \mathbf{u}_{\theta\mathrm{R}}$ or $\mathbf{h}_{\mathrm{R1}} = h_{\mathrm{R}} \, \mathbf{u}_{\varphi\mathrm{R}}$, where h_{R} is a nonzero complex number.

If $n = 2$, we assume that, in the direction $\theta' = \pi/2$ and $\varphi' = \pi$, the two antennas constituting the RAA are such that $\mathbf{Z}_{\mathrm{R}} = Z_{\mathrm{R}} \mathbf{1}_2$, and $\mathbf{h}_{\mathrm{R1}} = h_{\mathrm{R}} \, \mathbf{u}_{\theta\mathrm{R}}$ and $\mathbf{h}_{\mathrm{R2}} = h_{\mathrm{R}} \, \mathbf{u}_{\varphi\mathrm{R}}$, where Z_{R} and h_{R} are nonzero complex numbers, Z_{R} having a positive real part (since we assumed that $\mathbf{Z}_{\mathrm{R}}$ is positive definite). Thus, we have $\operatorname{rank} \mathbf{h}_{\mathrm{R}} = 2$. For instance, these antennas may be electrically short center-fed thin cylindrical dipole antennas, suitably positioned.

E. RECEPTION IN CONFIGURATION A

If the RAA is used for reception, an LTI multiport load having n ports, called MLR, is coupled to the RAA. The ports of the MLR are numbered from 1 to n, and, for any integer $q \in \{1, \dots, n\}$, port q of the MLR is connected to port q of the RAA (positive terminal to positive terminal and negative terminal to negative terminal). Let $\mathbf{V}_{\mathrm{AO2}}$ be the column vector of the rms open-circuit voltages at ports 1 to n of the RAA during CA. We find that

$$\mathbf{V}_{\mathrm{AO2}} = \mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{E}}_{\mathrm{AR}} \,, \tag{20}$$

where $\underset{\sim}{\mathbf{E}}_{\mathrm{AR}}$ is $\underset{\sim}{\mathbf{E}}_{\mathrm{A}}$ at point R in CA.

We assume that the MLR has an impedance matrix, equal to $\mathbf{Z}_{\mathrm{R}}^{*}$. It follows that the column vector of the rms currents flowing into ports 1 to n of the RAA used for reception, denoted by $\mathbf{I}_{\mathrm{A2}}$, is given by

$$\mathbf{I}_{\mathrm{A2}} = -\frac{1}{2\mathrm{Re}(Z_{\mathrm{R}})} \, \mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{E}}_{\mathrm{AR}} \,, \tag{21}$$

where $\mathrm{Re}(z)$ denotes the real part of a complex number z. The average power delivered by the ports of the RAA in CA, denoted by P_{ADP2}, and the available power of the RAA in CA, denoted by P_{AAVR}, are equal and given by

$$P_{\mathrm{ADP2}} = P_{\mathrm{AAVR}} = \mathrm{Re}(Z_{\mathrm{R}}) \, \mathbf{I}_{\mathrm{A2}}^{*} \mathbf{I}_{\mathrm{A2}} \,. \tag{22}$$

In the case $n = 1$, since $\mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{E}}_{\mathrm{AR}}$ is a complex number equal to $\mathbf{h}_{\mathrm{R1}} \cdot \mathbf{E}_{\mathrm{AR}}$, we get

$$P_{\mathrm{ADP2}} = P_{\mathrm{AAVR}} = \frac{1}{4\mathrm{Re}(Z_{\mathrm{R}})} \, |\mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{E}}_{\mathrm{AR}}|^2 \,. \tag{23}$$

It follows that

$$P_{\mathrm{ADP2}} = P_{\mathrm{AAVR}} = \frac{\mathbf{h}_{\mathrm{R}}^{*} \mathbf{h}_{\mathrm{R}}}{4\mathrm{Re}(Z_{\mathrm{R}})} \, e_{\mathrm{pol}}^{2} \, ||\mathbf{E}_{\mathrm{AR}}||^2 \,, \tag{24}$$

where

$$e_{\mathrm{pol}} = \frac{|\mathbf{h}_{\mathrm{R1}} \cdot \mathbf{E}_{\mathrm{AR}}|}{||\mathbf{h}_{\mathrm{R1}}|| \, ||\mathbf{E}_{\mathrm{AR}}||} \tag{25}$$

is the polarization mismatch factor of the RAA [13, Sec. 5.2], [14, Sec. 16.5]. A known result on e_{pol} is derived in Appendix A. By (24), in the case $n = 1$, $P_{\mathrm{ADP2}} = P_{\mathrm{AAVR}}$ is a measure of $e_{\mathrm{pol}}^{2} \, ||\mathbf{E}_{\mathrm{AR}}||^2$.

In the case $n = 2$, since

$$\mathbf{h}_{\mathrm{R}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{E}}_{\mathrm{AR}} = h_{\mathrm{R}} \begin{pmatrix} \mathbf{u}_{\theta\mathrm{R}} \cdot \mathbf{E}_{\mathrm{AR}} \\ \mathbf{u}_{\varphi\mathrm{R}} \cdot \mathbf{E}_{\mathrm{AR}} \end{pmatrix} , \tag{26}$$

in the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$, we get

$$P_{\mathrm{ADP2}} = P_{\mathrm{AAVR}} = \frac{|h_{\mathrm{R}}|^2}{4\mathrm{Re}(Z_{\mathrm{R}})} \, ||\mathbf{E}_{\mathrm{AR}}||^2 \,, \tag{27}$$

so that $P_{\mathrm{ADP2}} = P_{\mathrm{AAVR}}$ is a measure of $||\mathbf{E}_{\mathrm{AR}}||^2$.

F. EMISSION IN CONFIGURATION B

If the RAA is used for emission, an LTI multiport generator having n ports, called MGR, is coupled to the RAA. The ports of the MGR are numbered from 1 to n, and, for any integer $q \in \{1, \ldots, n\}$, port q of the MGR is connected to port q of the RAA (positive terminal to positive terminal and negative terminal to negative terminal). We assume that the MGR has an internal impedance matrix, equal to $\mathbf{Z}_{\mathrm{R}}^*$.

In CB, the excitation of the RAA may be defined using the column vector of the rms open-circuit voltages at ports 1 to n of the MGR, denoted by $\mathbf{V}_{\mathrm{BO2}}$, as variable. $\mathbf{V}_{\mathrm{BO2}}$ may take on any value lying in $\mathbb{C}^n$. The column vector of the rms currents flowing into ports 1 to n of the RAA, denoted by $\mathbf{I}_{\mathrm{B2}}$, can take on any value lying in $\mathbb{C}^n$ and is given by

$$\mathbf{I}_{\mathrm{B2}} = \frac{1}{2\mathrm{Re}(Z_{\mathrm{R}})} \mathbf{V}_{\mathrm{BO2}} \,. \tag{28}$$

The available power of the MGR in CB, denoted by P_{BAVG}, and the average power received by the ports of the RAA in CB, denoted by P_{BRP2}, are equal and given by

$$P_{\mathrm{BAVG}} = P_{\mathrm{BRP2}} = \mathrm{Re}(Z_{\mathrm{R}})\, \mathbf{I}_{\mathrm{B2}}^* \, \mathbf{I}_{\mathrm{B2}} \,. \tag{29}$$

The linearity of the RAA entails that

$$\mathbf{E}_{\mathrm{B0}} = j\eta \frac{k\, e^{-jkr_{\mathrm{R}}}}{4\pi r_{\mathrm{R}}} \sum_{q=1}^{n} I_{\mathrm{B2}\,q}\, \mathbf{h}_{\mathrm{R}q} \,, \tag{30}$$

where $I_{\mathrm{B2}\,1}, \ldots, I_{\mathrm{B2}\,n}$ are the entries of $\mathbf{I}_{\mathrm{B2}}$, and where $\mathbf{h}_{\mathrm{R1}}, \ldots, \mathbf{h}_{\mathrm{R}n}$ are considered in the direction $\theta' = \pi/2$ and $\varphi' = \pi$. In the derivation of (30), we have used a superposition of SPBO-1 to SPBO-n excited by the currents $I_{\mathrm{B2}\,1}$ to $I_{\mathrm{B2}\,n}$, respectively. This is possible because $\mathbf{I}_{\mathrm{B2}}$ may take on any value lying in $\mathbb{C}^n$. It follows that

$$\underset{\sim}{\mathbf{E}}_{\mathrm{B0}} = j\eta \frac{k\, e^{-jkr_{\mathrm{R}}}}{4\pi r_{\mathrm{R}}} \mathbf{h}_{\mathrm{R}} \mathbf{I}_{\mathrm{B2}} \,, \tag{31}$$

in which $\mathbf{h}_{\mathrm{R}}$ is considered in the direction $\theta' = \pi/2$ and $\varphi' = \pi$. Note that $\mathbf{E}_{\mathrm{B0}}$ and $\mathbf{h}_{\mathrm{R1}}, \ldots, \mathbf{h}_{\mathrm{R}n}$ are vectors of $\mathbb{E}^{\mathrm{c}}$ in (30). In contrast, in (31), $\underset{\sim}{\mathbf{E}}_{\mathrm{B0}}$ is a column vector and $\mathbf{h}_{\mathrm{R}}$ is a 3 by n matrix, both depending on the choice of an orthonormal basis of $\mathbb{E}$. If the chosen basis is $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$, then all entries of the first row of $\mathbf{h}_{\mathrm{R}}$ are zero.

If we consider CB in the case $n = 2$, we can use the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$ in (31), to obtain

$$\underset{\sim}{\mathbf{E}}_{\mathrm{B0}} = j\eta \frac{k\, e^{-jkr_{\mathrm{R}}}}{4\pi r_{\mathrm{R}}} \begin{pmatrix} 0 & 0 \\ h_{\mathrm{R}} & 0 \\ 0 & h_{\mathrm{R}} \end{pmatrix} \mathbf{I}_{\mathrm{B2}} \,, \tag{32}$$

which shows that all nonzero $\mathbf{I}_{\mathrm{B2}} \in \mathbb{C}^2$ cover all possible polarizations of $\mathbf{E}_{\mathrm{B0}}$.

IV. PARTIAL ABSOLUTE GAIN AND PARTIAL ABSOLUTE EFFECTIVE AREA

The partial gain of a single-port antenna, in a given direction, for a given wave polarization, is the ratio of a part of the radiation intensity produced by the antenna in the given direction to the radiation intensity that would be obtained if the (average) power accepted by the antenna were radiated equally in all directions, said part corresponding to the given polarization [1]–[2]. Consequently, we define the partial absolute gain of the MAA, in a given direction, for a given wave polarization and a specified nonzero excitation, as the ratio of a part of the radiation intensity produced by the MAA in the given direction to the radiation intensity that would be obtained if the average power received by the N ports of the MAA were radiated equally in all directions, said part corresponding to the given wave polarization. The given wave polarization is normally specified by a polarization vector, which is a dimensionless unit vector of $\mathbb{E}^{\mathrm{c}}$ of the given wave polarization (so that the polarization vector is orthogonal to the given direction).

It follows that the partial absolute gain of the MAA in the direction $\theta = \theta_{\mathrm{R}}$ and $\varphi = \varphi_{\mathrm{R}}$ for a polarization vector $\mathbf{u}_{\mathrm{pol}}$ and the specified nonzero excitation, denoted by $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, is given by

$$G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{4\pi r_{\mathrm{R}}^2\, |\mathbf{u}_{\mathrm{pol}} \cdot \mathbf{E}_{\mathrm{AR}}|^2}{\eta\, P_{\mathrm{ARP1}}} \,. \tag{33}$$

If we consider CA in the case $n = 1$, we can assume that the RAA is such that

$$\mathbf{u}_{\mathrm{pol}} = \frac{\mathbf{h}_{\mathrm{R1}}}{||\mathbf{h}_{\mathrm{R1}}||} \,, \tag{34}$$

so that (25) leads us to

$$G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{4\pi r_{\mathrm{R}}^2}{\eta\, P_{\mathrm{ARP1}}}\, e_{\mathrm{pol}}^2\, ||\mathbf{E}_{\mathrm{AR}}||^2 \,. \tag{35}$$

It follows from (24) and (35) that

$$G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{16\pi r_{\mathrm{R}}^2\, \mathrm{Re}(Z_{\mathrm{R}})}{\mathbf{h}_{\mathrm{R}}^* \mathbf{h}_{\mathrm{R}}\, \eta}\, g_{\mathrm{AU1}} \,, \tag{36}$$

where g_{AU1} is the unnamed power gain $P_{\mathrm{AAVR}}/P_{\mathrm{ARP1}}$ in CA in the case $n = 1$, for the excitation defined by $\mathbf{I}_{\mathrm{A1}}$.

The partial effective area of a single-port antenna, in a given direction, for a given wave polarization, is the ratio of the available power at the terminals of the antenna used for reception to the power flux density of a (uniform) plane wave of the given polarization incident on the antenna from the given direction [1]–[2] . This definition may be used without adaptation for the MAA: the partial absolute effective area of the MAA, in a given direction, for a given wave polarization, is the ratio of the available power at the N ports of the MAA used for reception to the power flux density of a uniform plane wave of the given polarization incident on the MAA from the given direction.

The given polarization being defined by a polarization vector $\mathbf{u}_{\mathrm{pol}}$, we consider CB in the case $n = 1$, and assume that the RAA is such that (34) is satisfied, so that the partial absolute effective area of the MAA in the direction $\theta = \theta_{\mathrm{R}}$ and $\varphi = \varphi_{\mathrm{R}}$ for a polarization vector $\mathbf{u}_{\mathrm{pol}}$, denoted by $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, is given by

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \eta \frac{P_{\mathrm{BAVA}}}{\underset{\sim}{\mathbf{E}}_{\mathrm{B0}}^* \underset{\sim}{\mathbf{E}}_{\mathrm{B0}}} \,. \tag{37}$$

Using (31) in (37) and the same assumptions, we get

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{4\lambda^2 r_{\mathrm{R}}^2}{\eta} \frac{P_{\mathrm{BAVA}}}{\mathbf{I}_{\mathrm{B2}}^*\, \mathbf{h}_{\mathrm{R}}^* \mathbf{h}_{\mathrm{R}}\, \mathbf{I}_{\mathrm{B2}}} \,, \tag{38}$$

where $\mathbf{I}_{\mathrm{B2}}$ is a nonzero scalar because $n = 1$. Thus, using (29), we obtain

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{4\lambda^2 r_{\mathrm{R}}^2 \, \mathrm{Re}(Z_{\mathrm{R}})}{\eta \, \mathbf{h}_{\mathrm{R}}^* \mathbf{h}_{\mathrm{R}}} \, g_{\mathrm{BU1}} \,, \tag{39}$$

where g_{BU1} is the unnamed power gain $P_{\mathrm{BAVA}}/P_{\mathrm{BRP2}}$ in CB in the case $n = 1$.

We observe that: (36) provides a connection between $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ and g_{AU1}, which are both determined for the specified nonzero excitation; and (39) provides a connection between $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ and g_{BU1}, which do not depend on an excitation. This will now allow us to obtain a relationship between $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ and $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, without computation.

If $N = 1$, that is, if the MAA is a single-port antenna, $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ does not depend on an arbitrary nonzero $\mathbf{I}_{\mathrm{A1}}$. Furthermore, if $N = 1$ and the MAA is a reciprocal single-port antenna, we have $g_{\mathrm{AU1}} = g_{\mathrm{BU1}}$ according to [3, Sec. IV.G], and it follows from (36) and (39) that

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\lambda^2}{4\pi} G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) \,, \tag{40}$$

which is a well-known result of antenna theory.

For any $N \geqslant 1$, we can observe that

$$\frac{G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})}{g_{\mathrm{AU1}}} = \frac{16\pi r_{\mathrm{R}}^2 \, \mathrm{Re}(Z_{\mathrm{R}})}{\mathbf{h}_{\mathrm{R}}^* \mathbf{h}_{\mathrm{R}} \, \eta} \tag{41}$$

is a positive real number, and use Theorem 29 of [6] and the explanations of Appendix B to assert that, in a given direction:

(a) the set of the values of $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, obtained for all nonzero $\mathbf{I}_{\mathrm{A1}}$, or equivalently for all nonzero $\mathbf{V}_{\mathrm{AO1}}$, has a least element referred to as "minimum partial absolute gain" and denoted by $G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}})$, and a greatest element referred to as "maximum partial absolute gain" and denoted by $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$; and

(b) if the MAA is reciprocal, then

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\lambda^2}{4\pi} G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}}) \,. \tag{42}$$

For practical computations of $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, we can use (16) and (18)–(19) in (37) to get

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\eta}{4} \frac{\underset{\sim}{\mathbf{E}}_{\mathrm{B0}}^* \overline{\mathbf{h}_{\mathrm{B}}} \, H(\mathbf{Z}_{\mathrm{A}})^{-1} \mathbf{h}_{\mathrm{B}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{E}}_{\mathrm{B0}}}{\underset{\sim}{\mathbf{E}}_{\mathrm{B0}}^* \underset{\sim}{\mathbf{E}}_{\mathrm{B0}}} \,, \tag{43}$$

where (34) is assumed to be satisfied, so that (31) leads us to

$$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\eta}{4} \, \underset{\sim}{\mathbf{u}}_{\mathrm{pol}}^* \overline{\mathbf{h}_{\mathrm{B}}} \, H(\mathbf{Z}_{\mathrm{A}})^{-1} \mathbf{h}_{\mathrm{B}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{u}}_{\mathrm{pol}} \,. \tag{44}$$

For practical computations of $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ in a context where the MAA is reciprocal, the simplest formula is

$$G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\pi\eta}{\lambda^2} \, \underset{\sim}{\mathbf{u}}_{\mathrm{pol}}^* \overline{\mathbf{h}_{\mathrm{A}}} \, H(\mathbf{Z}_{\mathrm{A}})^{-1} \mathbf{h}_{\mathrm{A}}^{\mathrm{T}} \, \underset{\sim}{\mathbf{u}}_{\mathrm{pol}} \,, \tag{45}$$

which directly follows from (42), (44) and $\mathbf{h}_{\mathrm{B}} = \mathbf{h}_{\mathrm{A}}$.

For other computations relating to $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, and in a context where the MAA need not be reciprocal, we can use (8) and (11) in (33) to get

$$G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\pi\eta \, |\underset{\sim}{\mathbf{u}}_{\mathrm{pol}}^{\mathrm{T}} (\mathbf{h}_{\mathrm{A}} \mathbf{I}_{\mathrm{A1}})|^2}{\lambda^2 \, \mathbf{I}_{\mathrm{A1}}^* \, H(\mathbf{Z}_{\mathrm{A}}) \, \mathbf{I}_{\mathrm{A1}}} \,. \tag{46}$$

This is a convenient expression to compute $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ for a given $\mathbf{I}_{\mathrm{A1}}$. To obtain $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ and $G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}})$, we write

$$G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\mathbf{I}_{\mathrm{A1}}^* \, \mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}}) \, \mathbf{I}_{\mathrm{A1}}}{\mathbf{I}_{\mathrm{A1}}^* \, H(\mathbf{Z}_{\mathrm{A}}) \, \mathbf{I}_{\mathrm{A1}}} \,. \tag{47}$$

where the N by N matrix

$$\mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\pi\eta}{\lambda^2} \, \mathbf{h}_{\mathrm{A}}^* \overline{\underset{\sim}{\mathbf{u}}_{\mathrm{pol}}} \underset{\sim}{\mathbf{u}}_{\mathrm{pol}}^{\mathrm{T}} \mathbf{h}_{\mathrm{A}} \tag{48}$$

is positive semidefinite because $|\underset{\sim}{\mathbf{u}}_{\mathrm{pol}}^{\mathrm{T}} (\mathbf{h}_{\mathrm{A}} \mathbf{I}_{\mathrm{A1}})|$ is nonnegative for any $\mathbf{I}_{\mathrm{A1}} \in \mathbb{C}^N$. Here, since $H(\mathbf{Z}_{\mathrm{A}})$ is positive definite, $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ is written as a generalized Rayleigh ratio of $\mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}})$ to $H(\mathbf{Z}_{\mathrm{A}})$, in the variable $\mathbf{I}_{\mathrm{A1}}$. It follows that we can use the theorem of Section II to obtain $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ and $G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}})$. Accordingly, the eigenvalues of $\mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}}) \, H(\mathbf{Z}_{\mathrm{A}})^{-1}$ are real and nonnegative, $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ is the largest of these eigenvalues, and $G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}})$ is the least of these eigenvalues.

We observe that $\underset{\sim}{\mathbf{u}}_{\mathrm{pol}} \underset{\sim}{\mathbf{u}}_{\mathrm{pol}}^*$ is a matrix of rank 1 because $\mathbf{u}_{\mathrm{pol}}$ is nonzero. This fact and (48) lead us to

$$\mathrm{rank}\, \mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}}) \leqslant 1 \,. \tag{49}$$

It follows from (49) that $\mathrm{rank}(\mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}}) H(\mathbf{Z}_{\mathrm{A}})^{-1}) \leqslant 1$, so that (47) and the theorem of Section II lead us to

$$(N = 1) \Longrightarrow (G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}}) = G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})) \tag{50}$$

and

$$(N > 1) \Longrightarrow (G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}}) = 0) \,. \tag{51}$$

Consequently, $G_{\mathrm{pa\,MIN}}(\mathbf{u}_{\mathrm{pol}})$ is not an interesting characteristic of the MAA, and was therefore not listed in Table 2.

It also follows from (47), (49) and the theorem of Section II that a mean value of $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ over a number N of linearly independent excitations is given by

$$G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}}) = \frac{G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})}{N} \tag{52}$$

and referred to as "mean partial absolute gain".

V. ABSOLUTE GAIN AND ABSOLUTE EFFECTIVE AREA

The absolute gain of a single-port antenna is the ratio of the radiation intensity produced in a given direction by the antenna to the radiation intensity that would be obtained if the (average) power accepted by the antenna were radiated equally in all directions [1]–[2]. Consequently, we define the absolute gain of the MAA, in a given direction, for a specified nonzero excitation, as the ratio of the radiation intensity produced by the MAA in the given direction to the radiation intensity that would be obtained if the average power received by the N ports of the MAA were radiated equally in all directions.

It follows that the absolute gain of the MAA in the direction $\theta = \theta_{\mathrm{R}}$ and $\varphi = \varphi_{\mathrm{R}}$ for the specified nonzero excitation, denoted by G_{a}, is given by

$$G_{\mathrm{a}} = \frac{4\pi r_{\mathrm{R}}^2}{\eta \, P_{\mathrm{ARP1}}} \, ||\mathbf{E}_{\mathrm{AR}}||^2 \,. \tag{53}$$

If we consider CA in the case $n = 2$, using (27), we get

$$G_{\rm a} = \frac{16\pi r_{\rm R}^2 \,{\rm Re}(Z_{\rm R})}{|h_{\rm R}|^2 \,\eta} \, g_{\rm AU2} \,, \tag{54}$$

where $g_{\rm AU2}$ is the unnamed power gain $P_{\rm AAVR}/P_{\rm ARP1}$ in CA in the case $n = 2$, for the excitation defined by $\mathbf{I}_{\rm A1}$.

The effective area (also called total effective area) of a single-port antenna, in a given direction, is the ratio of the available power at the terminals of the antenna used for reception to the power flux density of a plane wave incident on the antenna from the given direction, the wave being polarization-matched to the antenna [1]–[2]. This definition cannot be directly adapted to the MAA, because it need not be possible to assign a polarization to the MAA, so that the requirement "polarization-matched" is meaningless in this context. However, we may consider that, for a single-port antenna, the requirement "polarization-matched" corresponds to a polarization that maximizes the available power at the port of the antenna. Consequently, we define the absolute effective area of the MAA, in a given direction, as the ratio of the available power at the N ports of the MAA used for reception to the power flux density of a uniform plane wave incident on the MAA from the given direction, the polarization of the uniform plane wave being such that it maximizes this available power.

This particular polarization being defined by a polarization vector $\mathbf{u}_{\rm am}$, we consider CB in the case $n = 2$. Since $\mathbf{u}_{\rm am} \cdot \mathbf{u}_{r{\rm R}} = 0$, it follows from (32) that there exists a value $\mathbf{I}_{\rm B2\,am}$ of $\mathbf{I}_{\rm B2}$ that produces an incident field for which the polarization vector is $\mathbf{u}_{\rm am}$, that is to say, such that

$$\mathbf{u}_{\rm am} = \frac{\mathbf{E}_{\rm B0}}{||\mathbf{E}_{\rm B0}||} \,, \tag{55}$$

a suitable $\mathbf{I}_{\rm B2\,am}$ being, if we use the basis $(\mathbf{u}_{r{\rm R}}, \mathbf{u}_{\theta{\rm R}}, \mathbf{u}_{\varphi{\rm R}})$, given by

$$\mathbf{I}_{\rm B2\,am} = I_{\rm B2\,0} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{\sim}{\mathbf{u}}_{\rm am} \,, \tag{56}$$

where $I_{\rm B2\,0}$ is an arbitrary nonzero complex current, for instance equal to 1 A.

Thus, if $\mathbf{I}_{\rm B2} = \mathbf{I}_{\rm B2\,am}$, the absolute effective area of the MAA in the direction $\theta = \theta_{\rm R}$ and $\varphi = \varphi_{\rm R}$, denoted by $A_{\rm a}$, is given by

$$A_{\rm a} = \eta \, \frac{P_{\rm BAVA}}{\underset{\sim}{\mathbf{E}}^*_{\rm B0} \underset{\sim}{\mathbf{E}}_{\rm B0}} \,. \tag{57}$$

For a practical computation of $A_{\rm a}$, we can use the basis $(\mathbf{u}_{r{\rm R}}, \mathbf{u}_{\theta{\rm R}}, \mathbf{u}_{\varphi{\rm R}})$, (16), (18)–(19) and (32) in (57) to obtain

$$A_{\rm a} = \frac{\eta}{4} \, \frac{\mathbf{I}^*_{\rm B2\,am} \mathbf{P}^* \overline{\mathbf{h}_{\rm B}} \, H(\mathbf{Z}_{\rm A})^{-1} \mathbf{h}_{\rm B}^{\rm T} \, \mathbf{P} \mathbf{I}_{\rm B2\,am}}{\mathbf{I}^*_{\rm B2\,am} \mathbf{I}_{\rm B2\,am}} \,, \tag{58}$$

where

$$\mathbf{P} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{59}$$

It follows that $A_{\rm a}$ can be computed using Rayleigh's theorem [7, Sec. 4.2.2], as the maximum value of the Rayleigh ratio

$$r_{\rm a}(\mathbf{I}_{\rm B2}) = \frac{\mathbf{I}^*_{\rm B2} \mathbf{N}_{\rm Ba} \, \mathbf{I}_{\rm B2}}{\mathbf{I}^*_{\rm B2} \, \mathbf{I}_{\rm B2}} \tag{60}$$

in the variable $\mathbf{I}_{\rm B2}$, where, using $\mathbf{h}_{\rm B}$ determined in the basis $(\mathbf{u}_{r{\rm R}}, \mathbf{u}_{\theta{\rm R}}, \mathbf{u}_{\varphi{\rm R}})$, $\mathbf{N}_{\rm Ba}$ is a 2 by 2 matrix given by

$$\mathbf{N}_{\rm Ba} = \frac{\eta}{4} \, \mathbf{P}^{\rm T} \, \overline{\mathbf{h}_{\rm B}} \, H(\mathbf{Z}_{\rm A})^{-1} \mathbf{h}_{\rm B}^{\rm T} \, \mathbf{P} \,, \tag{61}$$

which is positive semidefinite because $H(\mathbf{Z}_{\rm A})^{-1}$ is positive definite. Accordingly, the eigenvalues of $\mathbf{N}_{\rm Ba}$ are real and nonnegative, and $A_{\rm a}$ is the largest of these eigenvalues.

By (15) and (61), we have

$$\operatorname{rank} \mathbf{N}_{\rm Ba} \leqslant \min\{N, 2\} \,. \tag{62}$$

Using the basis $(\mathbf{u}_{r{\rm R}}, \mathbf{u}_{\theta{\rm R}}, \mathbf{u}_{\varphi{\rm R}})$, and (32) in (57), we get

$$A_{\rm a} = \frac{4\lambda^2 r_{\rm R}^2}{\eta \, |h_{\rm R}|^2} \, \frac{P_{\rm BAVA}}{\mathbf{I}^*_{\rm B2\,am} \mathbf{I}_{\rm B2\,am}} \,, \tag{63}$$

which, by (29) leads us to

$$A_{\rm a} = \frac{4\lambda^2 r_{\rm R}^2 \,{\rm Re}(Z_{\rm R})}{\eta \, |h_{\rm R}|^2} \, g_{\rm BU2\,am} \,, \tag{64}$$

where $g_{\rm BU2\,am}$ is the unnamed power gain $P_{\rm BAVA}/P_{\rm BRP2}$ in CB in the case $n = 2$, if $\mathbf{I}_{\rm B2} = \mathbf{I}_{\rm B2\,am}$.

Let $g_{\rm BU2}$ be the unnamed power gain $P_{\rm BAVA}/P_{\rm BRP2}$ in CB in the case $n = 2$, for a given $\mathbf{I}_{\rm B2}$ that need not be equal to $\mathbf{I}_{\rm B2\,am}$. We observe that: (54) provides a connection between $G_{\rm a}$ and $g_{\rm AU2}$, which are both determined for a specified nonzero excitation; and (64) provides a connection between $A_{\rm a}$ and $g_{\rm BU2\,am}$, which are both determined for an excitation that maximizes $g_{\rm BU2}$. This will now allow us to get a relationship between $G_{\rm a}$ and $A_{\rm a}$, without computation.

It follows from (54) and (64) that $G_{\rm a}/g_{\rm AU2}$ and $A_{\rm a}/g_{\rm BU2\,am}$ are positive real numbers. We can therefore use Theorem 29 of [6] and the explanations of Appendix B to assert that, in a given direction:

(a) the set of the values of $G_{\rm a}$, obtained for all nonzero $\mathbf{I}_{\rm A1}$, or equivalently for all nonzero $\mathbf{V}_{\rm AO1}$, has a least element referred to as "minimum absolute gain" and denoted by $G_{\rm a\,MIN}$, and a greatest element referred to as "maximum absolute gain" and denoted by $G_{\rm a\,MAX}$; and
(b) if the MAA is reciprocal, then

$$A_{\rm a} = \frac{\lambda^2}{4\pi} G_{\rm a\,MAX} \,. \tag{65}$$

If $N = 1$, that is, if the MAA is a single-port antenna, $G_{\rm a}$ does not depend on an arbitrary nonzero $\mathbf{I}_{\rm A1}$. Furthermore, if $N = 1$ and the MAA is a reciprocal single-port antenna, it follows from (65) that we have

$$A_{\rm a} = \frac{\lambda^2}{4\pi} G_{\rm a} \,, \tag{66}$$

which is a well-known result of antenna theory.

For any $N \geqslant 1$, in a context where the MAA need not be reciprocal, we can use (8) and (11) to obtain

$$G_{\rm a} = \frac{\mathbf{I}^*_{\rm A1} \mathbf{N}_{\rm A} \, \mathbf{I}_{\rm A1}}{\mathbf{I}^*_{\rm A1} \, H(\mathbf{Z}_{\rm A}) \, \mathbf{I}_{\rm A1}} \tag{67}$$

where the N by N matrix

$$\mathbf{N}_{\rm A} = \frac{\pi \eta}{\lambda^2} \, \mathbf{h}_{\rm A}^* \mathbf{h}_{\rm A} \tag{68}$$

is positive semidefinite by [7, Sec. 2.6.3] or [8, Sec. 7.3.1]. Here, since $H(\mathbf{Z}_A)$ is positive definite, G_a is written as a generalized Rayleigh ratio of $\mathbf{N}_A$ to $H(\mathbf{Z}_A)$, in the variable $\mathbf{I}_{A1}$. It follows that we can use the theorem of Section II to obtain $G_{a\,MAX}$ and $G_{a\,MIN}$. Accordingly, the eigenvalues of $\mathbf{N}_A H(\mathbf{Z}_A)^{-1}$ are real and nonnegative, $G_{a\,MAX}$ is the largest of these eigenvalues, and $G_{a\,MIN}$ is the least of these eigenvalues.

By (12) and (68), we have

$$\operatorname{rank} \mathbf{N}_A \leqslant \min\{N, 2\}\,. \tag{69}$$

It follows that the theorem of Section II leads us to

$$(N > 2) \Longrightarrow (G_{a\,MIN} = 0)\ . \tag{70}$$

VI. MINIMUM ABSOLUTE EQUIVALENT AREA AND MEAN ABSOLUTE EQUIVALENT AREA

We have discussed the partial absolute effective area of the MAA, in Section IV, using the case $n = 1$, and we have discussed the absolute effective area of the MAA, in Section V, using the case $n = 2$. Let us now study the partial absolute effective area of the MAA, in the case $n = 2$, for an arbitrary polarization vector $\mathbf{u}_{pol}$. Since $\mathbf{u}_{pol} \cdot \mathbf{u}_{rR} = 0$, it follows from (32) that there exists a value $\mathbf{I}_{B2\,pol}$ of $\mathbf{I}_{B2}$ that produces an incident field for which the polarization vector is $\mathbf{u}_{pol}$, that is to say, such that

$$\frac{\mathbf{E}_{B0}}{||\mathbf{E}_{B0}||} = \mathbf{u}_{pol}\,, \tag{71}$$

a suitable $\mathbf{I}_{B2\,pol}$ being, if we use the basis $(\mathbf{u}_{rR}, \mathbf{u}_{\theta R}, \mathbf{u}_{\varphi R})$, given by

$$\mathbf{I}_{B2\,pol} = I_{B2\,0} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{\sim}{\mathbf{u}}_{pol}\,, \tag{72}$$

where $I_{B2\,0}$ is an arbitrary nonzero complex current, for instance equal to 1 A. Thus, if $\mathbf{I}_{B2} = \mathbf{I}_{B2\,pol}$, we have

$$A_{pa}(\mathbf{u}_{pol}) = \eta \frac{P_{BAVA}}{\underset{\sim}{\mathbf{E}}_{B0}^{*}\underset{\sim}{\mathbf{E}}_{B0}} \tag{73}$$

in the direction $\theta = \theta_R$ and $\varphi = \varphi_R$.

For a practical computation of $A_{pa}(\mathbf{u}_{pol})$, we can use the basis $(\mathbf{u}_{rR}, \mathbf{u}_{\theta R}, \mathbf{u}_{\varphi R})$, (16), (18)–(19) and (32) in (73) to obtain

$$A_{pa}(\mathbf{u}_{pol}) = \frac{\eta}{4} \frac{\mathbf{I}_{B2\,pol}^{*}\mathbf{P}^{*}\overline{\mathbf{h}_B}\, H(\mathbf{Z}_A)^{-1}\mathbf{h}_B^{T}\,\mathbf{P}\mathbf{I}_{B2\,pol}}{\mathbf{I}_{B2\,pol}^{*}\mathbf{I}_{B2\,pol}}\,. \tag{74}$$

It follows that, in this basis,

$$A_{pa}(\mathbf{u}_{pol}) = r_a(\mathbf{I}_{B2\,pol})\,, \tag{75}$$

where r_a is the Rayleigh ratio defined by (60).

We define the minimum absolute equivalent area of the MAA, in a given direction, as the ratio of the available power at the N ports of the MAA used for reception to the power flux density of a uniform plane wave incident on the MAA from the given direction, the polarization of the uniform plane wave being such that it minimizes this available power.

Thus, the minimum absolute equivalent area of the MAA in the direction $\theta = \theta_R$ and $\varphi = \varphi_R$, denoted by $A_{aeq\,MIN}$, is the minimum value of the Rayleigh ratio r_a, which is the least eigenvalue of $\mathbf{N}_{Ba}$ according to Rayleigh's theorem.

It follows from (62) that at most $\min\{N, 2\}$ eigenvalues of $\mathbf{N}_{Ba}$ are nonzero. The sum of these eigenvalues being $\operatorname{tr} \mathbf{N}_{Ba}$, we define the mean absolute equivalent area of the MAA, in a given direction, as

$$A_{aeq\,MEA} = \frac{\operatorname{tr} \mathbf{N}_{Ba}}{2} = \frac{A_a + A_{aeq\,MIN}}{2}\,, \tag{76}$$

because it follows from (32), (71) and (75) that this quantity is a mean value of $A_{pa}(\mathbf{u}_{pol})$ over two orthogonal polarization vectors.

It follows from (62) that

$$(N = 1) \Rightarrow \left(A_{aeq\,MEA} = \frac{A_a}{2} \text{ and } A_{aeq\,MIN} = 0 \right). \tag{77}$$

Using the basis $(\mathbf{u}_{rR}, \mathbf{u}_{\theta R}, \mathbf{u}_{\varphi R})$, (32) in (73) and the assumption that $n = 2$ and $\mathbf{I}_{B2} = \mathbf{I}_{B2\,pol}$, we get

$$A_{pa}(\mathbf{u}_{pol}) = \frac{4\lambda^2 r_R^2}{\eta\,|h_R|^2}\,\frac{P_{BAVA}}{\mathbf{I}_{B2\,pol}^{*}\,\mathbf{I}_{B2\,pol}}\,, \tag{78}$$

and using (29) leads us to

$$A_{pa}(\mathbf{u}_{pol}) = \frac{4\lambda^2 r_R^2 \operatorname{Re}(Z_R)}{\eta\,|h_R|^2}\,g_{BU2\,pol}\,, \tag{79}$$

where $g_{BU2\,pol}$ is the unnamed power gain P_{BAVA}/P_{BRP2} in CB in the case $n = 2$, if $\mathbf{I}_{B2} = \mathbf{I}_{B2\,pol}$.

We noted in Section V that (54) provides a connection between G_a and g_{AU2}, which are both determined for a specified nonzero excitation. We now observe that (79) provides a connection between $A_{pa}(\mathbf{u}_{pol})$ and $g_{BU2\,pol}$, which is determined for an excitation that is given by (72) as a function of the arbitrary polarization vector $\mathbf{u}_{pol}$. This will now allow us to obtain two results relating to G_a and $A_{pa}(\mathbf{u}_{pol})$, without computation.

By (54) and (79), G_a/g_{AU2} and $A_{pa}(\mathbf{u}_{pol})/g_{BU2\,pol}$ are positive real numbers. It follows that we can use Theorem 29 of [6], (67), (75)–(76) and the explanations of Appendix B to assert that, in a given direction:

(a) if $\mathbf{M}$ is given by the theorem of Section II applied to the generalized Rayleigh ratio of $\mathbf{N}_A$ to $H(\mathbf{Z}_A)$, in the variable $\mathbf{I}_{A1}$, a mean value of G_a over a number N of linearly independent $\mathbf{I}_{A1}$ is referred to as "mean absolute gain" and given by

$$G_{a\,MEA} = \frac{\operatorname{tr} \mathbf{M}}{N} = \frac{\operatorname{tr}\left(\mathbf{N}_A H(\mathbf{Z}_A)^{-1}\right)}{N}\,; \tag{80}$$

and

(b) if the MAA is reciprocal, then

$$A_{aeq\,MEA} = \frac{N\lambda^2}{8\pi} G_{a\,MEA} \tag{81}$$

and

$$(N = 2) \Longrightarrow \left(A_{aeq\,MIN} = \frac{\lambda^2}{4\pi} G_{a\,MIN} \right). \tag{82}$$

VII. PARTIAL REACHED GAIN AND PARTIAL REACHED EFFECTIVE AREA

We define the partial reached gain of the MAA, in a given direction, for a given wave polarization and a specified nonzero excitation, as the ratio of a part of the radiation intensity produced by the MAA in the given direction to the radiation intensity that would be obtained if the available power of the multiport generator coupled to the N ports of the MAA were radiated equally in all directions, said part corresponding to the given wave polarization.

It follows that the partial reached gain of the MAA in the direction $\theta = \theta_{\text{R}}$ and $\varphi = \varphi_{\text{R}}$ for a polarization vector $\mathbf{u}_{\text{pol}}$ and the specified nonzero excitation, denoted by $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$, is given by

$$G_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{4\pi r_{\text{R}}^2 \, |\mathbf{u}_{\text{pol}} \cdot \mathbf{E}_{\text{AR}}|^2}{\eta \, P_{\text{AAVG}}} \,. \tag{83}$$

If we consider CA in the case $n = 1$, we can assume that the RAA is such that (34) is satisfied, so that (25) leads us to

$$G_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{4\pi r_{\text{R}}^2}{\eta \, P_{\text{AAVG}}} \, e_{\text{pol}}^2 \, ||\mathbf{E}_{\text{AR}}||^2 \,. \tag{84}$$

It follows from (24) and (84) that

$$G_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{16\pi r_{\text{R}}^2 \, \text{Re}(Z_{\text{R}})}{\mathbf{h}_{\text{R}}^* \mathbf{h}_{\text{R}} \; \eta} \, g_{\text{AT1}} \,, \tag{85}$$

where g_{AT1} is the transducer power gain $P_{\text{ADP2}}/P_{\text{AAVG}}$ in CA in the case $n = 1$, for the excitation defined by $\mathbf{I}_{\text{A1}}$.

We define the partial reached effective area of the MAA, in a given direction, for a given wave polarization, as the ratio of the average power delivered by the N ports of the MAA used for reception to the power flux density of a uniform plane wave of the given polarization incident on the MAA from the given direction.

The given polarization being defined by a polarization vector $\mathbf{u}_{\text{pol}}$, we consider CB in the case $n = 1$, and assume that the RAA is such that (34) is satisfied, so that the partial reached effective area of the MAA in the direction $\theta = \theta_{\text{R}}$ and $\varphi = \varphi_{\text{R}}$ for a polarization vector $\mathbf{u}_{\text{pol}}$, denoted by $A_{\text{pr}}(\mathbf{u}_{\text{pol}})$, is given by

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \eta \, \frac{P_{\text{BDP1}}}{\underset{\sim}{\mathbf{E}}_{\text{B0}}^* \underset{\sim}{\mathbf{E}}_{\text{B0}}} \,. \tag{86}$$

Using (31) in (86) and the same assumptions, we get

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{4\lambda^2 r_{\text{R}}^2}{\eta} \, \frac{P_{\text{BDP1}}}{\mathbf{I}_{\text{B2}}^* \, \mathbf{h}_{\text{R}}^* \mathbf{h}_{\text{R}} \, \mathbf{I}_{\text{B2}}} \,, \tag{87}$$

where $\mathbf{I}_{\text{B2}}$ is a nonzero scalar because $n = 1$. Thus, using (29), we obtain

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{4\lambda^2 r_{\text{R}}^2 \, \text{Re}(Z_{\text{R}})}{\eta \, \mathbf{h}_{\text{R}}^* \mathbf{h}_{\text{R}}} \, g_{\text{BT1}} \,, \tag{88}$$

where g_{BT1} is the transducer power gain $P_{\text{BDP1}}/P_{\text{BAVG}}$ in CB in the case $n = 1$.

We observe that: (85) provides a connection between $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$ and g_{AT1}, which are both determined for the specified nonzero excitation; and (88) provides a connection between $A_{\text{pr}}(\mathbf{u}_{\text{pol}})$ and g_{BT1}, which do not depend on an excitation. This will now allow us to obtain a relationship between $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$ and $A_{\text{pr}}(\mathbf{u}_{\text{pol}})$, without computation.

If $N = 1$, that is, if the MAA is a single-port antenna, $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$ does not depend on an arbitrary nonzero $\mathbf{I}_{\text{A1}}$. Furthermore, if $N = 1$ and the MAA is a reciprocal single-port antenna, we have $g_{\text{AT1}} = g_{\text{BT1}}$ according to [3, Sec. IV.C], and it follows from (85) and (88) that

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{\lambda^2}{4\pi} G_{\text{pr}}(\mathbf{u}_{\text{pol}}) \,. \tag{89}$$

For any $N \geqslant 1$, we can observe that

$$\frac{G_{\text{pr}}(\mathbf{u}_{\text{pol}})}{g_{\text{AT1}}} = \frac{16\pi r_{\text{R}}^2 \, \text{Re}(Z_{\text{R}})}{\mathbf{h}_{\text{R}}^* \mathbf{h}_{\text{R}} \; \eta} \tag{90}$$

is a positive real number, and use Theorem 4 of [5] or Theorem 15 of [6] to assert that, in a given direction:

(a) the set of the values of $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$, obtained for all nonzero $\mathbf{I}_{\text{A1}}$, or equivalently for all nonzero $\mathbf{V}_{\text{AO1}}$, has a least element referred to as "minimum partial reached gain" and denoted by $G_{\text{pr MIN}}(\mathbf{u}_{\text{pol}})$, and a greatest element referred to as "maximum partial reached gain" and denoted by $G_{\text{pr MAX}}(\mathbf{u}_{\text{pol}})$; and

(b) if the MAA is reciprocal and $\mathbf{Z}_{\text{G}}$ is symmetric, then

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{\lambda^2}{4\pi} G_{\text{pr MAX}}(\mathbf{u}_{\text{pol}}) \,. \tag{91}$$

For practical computations of $A_{\text{pr}}(\mathbf{u}_{\text{pol}})$, we can use (16)-(17) in (86) to get

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \eta \, \frac{\underset{\sim}{\mathbf{E}}_{\text{B0}}^* \overline{\mathbf{h}_{\text{B}}} \, \mathbf{Y}_{\text{AG}}^* H(\mathbf{Z}_{\text{G}}) \mathbf{Y}_{\text{AG}} \mathbf{h}_{\text{B}}^{\text{T}} \, \underset{\sim}{\mathbf{E}}_{\text{B0}}}{\underset{\sim}{\mathbf{E}}_{\text{B0}}^* \underset{\sim}{\mathbf{E}}_{\text{B0}}} \,, \tag{92}$$

where $\mathbf{Y}_{\text{AG}} = (\mathbf{Z}_{\text{A}} + \mathbf{Z}_{\text{G}})^{-1}$. Since (34) is assumed to be satisfied, (31) leads us to

$$A_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \eta \, \underset{\sim}{\mathbf{u}}_{\text{pol}}^* \overline{\mathbf{h}_{\text{B}}} \, \mathbf{Y}_{\text{AG}}^* H(\mathbf{Z}_{\text{G}}) \mathbf{Y}_{\text{AG}} \mathbf{h}_{\text{B}}^{\text{T}} \, \underset{\sim}{\mathbf{u}}_{\text{pol}} \,. \tag{93}$$

For practical computations of $G_{\text{pr MAX}}(\mathbf{u}_{\text{pol}})$ in a context where the MAA is reciprocal and $\mathbf{Z}_{\text{G}}$ is symmetric, the simplest formula is

$$G_{\text{pr MAX}}(\mathbf{u}_{\text{pol}}) = \frac{4\pi\eta}{\lambda^2} \, \underset{\sim}{\mathbf{u}}_{\text{pol}}^* \overline{\mathbf{h}_{\text{A}}} \, \mathbf{Y}_{\text{AG}}^* H(\mathbf{Z}_{\text{G}}) \mathbf{Y}_{\text{AG}} \mathbf{h}_{\text{A}}^{\text{T}} \, \underset{\sim}{\mathbf{u}}_{\text{pol}} \,, \tag{94}$$

which directly follows from (91), (93) and $\mathbf{h}_{\text{B}} = \mathbf{h}_{\text{A}}$.

For other computations relating to $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$, and in a context where the MAA need not be reciprocal and/or $\mathbf{Z}_{\text{G}}$ need not be symmetric, we can use (6) and (11) in (83) to get

$$G_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{\pi\eta \, |\underset{\sim}{\mathbf{u}}_{\text{pol}}^{\text{T}} (\mathbf{h}_{\text{A}} \mathbf{I}_{\text{A1}})|^2}{\lambda^2 \, \mathbf{I}_{\text{A1}}^* \, \mathbf{Z}_{\text{AAVGO}} \, \mathbf{I}_{\text{A1}}} \,, \tag{95}$$

which is convenient to compute $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$ for a given $\mathbf{I}_{\text{A1}}$. To get $G_{\text{pr MAX}}(\mathbf{u}_{\text{pol}})$ and $G_{\text{pr MIN}}(\mathbf{u}_{\text{pol}})$, we write

$$G_{\text{pr}}(\mathbf{u}_{\text{pol}}) = \frac{\mathbf{I}_{\text{A1}}^* \, \mathbf{N}_{\text{Ap}}(\mathbf{u}_{\text{pol}}) \, \mathbf{I}_{\text{A1}}}{\mathbf{I}_{\text{A1}}^* \, \mathbf{Z}_{\text{AAVGO}} \, \mathbf{I}_{\text{A1}}} \,. \tag{96}$$

where $\mathbf{N}_{\text{Ap}}(\mathbf{u}_{\text{pol}})$ is given by (48).

Since $\mathbf{Z}_{\text{AAVGO}}$ is positive definite, $G_{\text{pr}}(\mathbf{u}_{\text{pol}})$ is written as a generalized Rayleigh ratio of $\mathbf{N}_{\text{Ap}}(\mathbf{u}_{\text{pol}})$ to $\mathbf{Z}_{\text{AAVGO}}$, in the variable $\mathbf{I}_{\text{A1}}$. Consequently, we can use the theorem of Section II to obtain $G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ and $G_{\mathrm{pr\,MIN}}(\mathbf{u}_{\mathrm{pol}})$. Thus, the eigenvalues of $\mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}})\,\mathbf{Z}_{\mathrm{AAVGO}}^{-1}$ are real and nonnegative, $G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ is the largest of these eigenvalues, and $G_{\mathrm{pr\,MIN}}(\mathbf{u}_{\mathrm{pol}})$ is the least of these eigenvalues.

It follows from (49) that $\mathrm{rank}(\mathbf{N}_{\mathrm{Ap}}(\mathbf{u}_{\mathrm{pol}})\mathbf{Z}_{\mathrm{AAVGO}}^{-1}) \leqslant 1$, so that (96) and the theorem of Section II lead us to

$$(N=1) \Longrightarrow (G_{\mathrm{pr\,MIN}}(\mathbf{u}_{\mathrm{pol}}) = G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})) \tag{97}$$

and

$$(N>1) \Longrightarrow (G_{\mathrm{pr\,MIN}}(\mathbf{u}_{\mathrm{pol}}) = 0)\,. \tag{98}$$

Consequently, $G_{\mathrm{pr\,MIN}}(\mathbf{u}_{\mathrm{pol}})$ is not an interesting characteristic of the MAA, and was therefore not listed in Table 2.

It also follows from (49), (96) and the theorem of Section II that a mean value of $G_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})$ over a number N of linearly independent excitations is given by

$$G_{\mathrm{pr\,MEA}}(\mathbf{u}_{\mathrm{pol}}) = \frac{G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})}{N} \tag{99}$$

and referred to as "mean partial reached gain".

VIII. REACHED GAIN AND REACHED EFFECTIVE AREA

We define the reached gain of the MAA, in a given direction, for a specified nonzero excitation, as the ratio of the radiation intensity produced by the MAA in the given direction to the radiation intensity that would be obtained if the available power of the multiport generator coupled to the N ports of the MAA were radiated equally in all directions.

It follows that the reached gain of the MAA in the direction $\theta = \theta_{\mathrm{R}}$ and $\varphi = \varphi_{\mathrm{R}}$ for the specified nonzero excitation, denoted by G_{r}, is given by

$$G_{\mathrm{r}} = \frac{4\pi r_{\mathrm{R}}^2}{\eta\, P_{\mathrm{AAVG}}}\, ||\mathbf{E}_{\mathrm{AR}}||^2\,. \tag{100}$$

If we consider CA in the case $n=2$, using (27), we get

$$G_{\mathrm{r}} = \frac{16\pi r_{\mathrm{R}}^2\,\mathrm{Re}(Z_{\mathrm{R}})}{|h_{\mathrm{R}}|^2\,\eta}\, g_{\mathrm{AT2}}\,, \tag{101}$$

where g_{AT2} is the transducer power gain $P_{\mathrm{ADP2}}/P_{\mathrm{AAVG}}$ in CA in the case $n=2$, for the excitation defined by $\mathbf{I}_{\mathrm{A1}}$.

We define the reached effective area of the MAA, in a given direction, as the ratio of the average power delivered by the N ports of the MAA used for reception to the power flux density of a uniform plane wave incident on the MAA from the given direction, the polarization of the uniform plane wave being such that it maximizes this average power.

This particular polarization being defined by a polarization vector $\mathbf{u}_{\mathrm{rm}}$, we consider CB in the case $n=2$. Since $\mathbf{u}_{\mathrm{rm}} \cdot \mathbf{u}_{r\mathrm{R}} = 0$, it follows from (32) that there exists a value $\mathbf{I}_{\mathrm{B2\,rm}}$ of $\mathbf{I}_{\mathrm{B2}}$ that produces an incident field for which the polarization vector is $\mathbf{u}_{\mathrm{rm}}$, that is to say, such that

$$\mathbf{u}_{\mathrm{rm}} = \frac{\mathbf{E}_{\mathrm{B0}}}{||\mathbf{E}_{\mathrm{B0}}||}\,, \tag{102}$$

a suitable $\mathbf{I}_{\mathrm{B2\,rm}}$ being, if we use the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$, given by

$$\mathbf{I}_{\mathrm{B2\,rm}} = I_{\mathrm{B2\,0}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{\sim}{\mathbf{u}}_{\mathrm{rm}}\,, \tag{103}$$

where $I_{\mathrm{B2\,0}}$ is an arbitrary nonzero complex current, for instance equal to 1 A.

Thus, if $\mathbf{I}_{\mathrm{B2}} = \mathbf{I}_{\mathrm{B2\,rm}}$, the reached effective area of the MAA in the direction $\theta = \theta_{\mathrm{R}}$ and $\varphi = \varphi_{\mathrm{R}}$, denoted by A_{r}, is given by

$$A_{\mathrm{r}} = \eta\, \frac{P_{\mathrm{BDP1}}}{\underset{\sim}{\mathbf{E}}_{\mathrm{B0}}^{*}\underset{\sim}{\mathbf{E}}_{\mathrm{B0}}}\,. \tag{104}$$

For a practical computation of A_{r}, we can use the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$, (16)-(17) and (32) in (104) to obtain

$$A_{\mathrm{r}} = \eta\, \frac{\mathbf{I}_{\mathrm{B2\,rm}}^{*}\mathbf{P}^{*}\overline{\mathbf{h}_{\mathrm{B}}}\,\mathbf{Y}_{\mathrm{AG}}^{*}H(\mathbf{Z}_{\mathrm{G}})\mathbf{Y}_{\mathrm{AG}}\mathbf{h}_{\mathrm{B}}^{\mathrm{T}}\,\mathbf{P}\mathbf{I}_{\mathrm{B2\,rm}}}{\mathbf{I}_{\mathrm{B2\,rm}}^{*}\mathbf{I}_{\mathrm{B2\,rm}}}\,. \tag{105}$$

It follows that A_{r} can be computed using Rayleigh's theorem, as the maximum value of the Rayleigh ratio

$$r_{\mathrm{r}}(\mathbf{I}_{\mathrm{B2}}) = \frac{\mathbf{I}_{\mathrm{B2}}^{*}\mathbf{N}_{\mathrm{Br}}\,\mathbf{I}_{\mathrm{B2}}}{\mathbf{I}_{\mathrm{B2}}^{*}\,\mathbf{I}_{\mathrm{B2}}} \tag{106}$$

in the variable $\mathbf{I}_{\mathrm{B2}}$, where, using $\mathbf{h}_{\mathrm{B}}$ determined in the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$, $\mathbf{N}_{\mathrm{Br}}$ is a 2 by 2 matrix given by

$$\mathbf{N}_{\mathrm{Br}} = \eta\, \mathbf{P}^{\mathrm{T}}\,\overline{\mathbf{h}_{\mathrm{B}}}\,\mathbf{Y}_{\mathrm{AG}}^{*}H(\mathbf{Z}_{\mathrm{G}})\mathbf{Y}_{\mathrm{AG}}\mathbf{h}_{\mathrm{B}}^{\mathrm{T}}\,\mathbf{P}\,, \tag{107}$$

which is positive semidefinite because $\mathbf{Y}_{\mathrm{AG}}^{*}H(\mathbf{Z}_{\mathrm{G}})\mathbf{Y}_{\mathrm{AG}}$ is positive definite. Thus, the eigenvalues of $\mathbf{N}_{\mathrm{Br}}$ are real and nonnegative, and A_{r} is the largest of these eigenvalues.

By (15) and (107), we have

$$\mathrm{rank}\,\mathbf{N}_{\mathrm{Br}} \leqslant \min\{N, 2\}\,. \tag{108}$$

Using the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$, and (32) in (104), we get

$$A_{\mathrm{r}} = \frac{4\lambda^2 r_{\mathrm{R}}^2}{\eta\,|h_{\mathrm{R}}|^2}\, \frac{P_{\mathrm{BDP1}}}{\mathbf{I}_{\mathrm{B2\,rm}}^{*}\,\mathbf{I}_{\mathrm{B2\,rm}}}\,, \tag{109}$$

which, by (29) leads us to

$$A_{\mathrm{r}} = \frac{4\lambda^2 r_{\mathrm{R}}^2\,\mathrm{Re}(Z_{\mathrm{R}})}{\eta\,|h_{\mathrm{R}}|^2}\, g_{\mathrm{BT2\,rm}}\,, \tag{110}$$

where $g_{\mathrm{BT2\,rm}}$ is the transducer power gain $P_{\mathrm{BDP1}}/P_{\mathrm{BAVG}}$ in CB in the case $n=2$, if $\mathbf{I}_{\mathrm{B2}} = \mathbf{I}_{\mathrm{B2\,rm}}$.

Let g_{BT2} be the transducer power gain $P_{\mathrm{BDP1}}/P_{\mathrm{BAVG}}$ in CB in the case $n=2$, for a given $\mathbf{I}_{\mathrm{B2}}$ that need not be equal to $\mathbf{I}_{\mathrm{B2\,rm}}$. We observe that: (101) provides a connection between G_{r} and g_{AT2}, which are both determined for a specified nonzero excitation; and (110) provides a connection between A_{r} and $g_{\mathrm{BT2\,rm}}$, which are both determined for an excitation that maximizes g_{BT2}. This will now allow us to get a relationship between G_{r} and A_{r}, without computation.

Since (101) and (110) entail that $G_{\mathrm{r}}/g_{\mathrm{AT2}}$ and $A_{\mathrm{r}}/g_{\mathrm{BT2\,rm}}$ are positive real numbers, we can use Theorem 4 of [5] or Theorem 15 of [6] to assert that, in a given direction:

(a) the set of the values of G_{r}, obtained for all nonzero $\mathbf{I}_{\mathrm{A1}}$, or equivalently for all nonzero $\mathbf{V}_{\mathrm{AO1}}$, has a least element referred to as "minimum reached gain" and denoted by $G_{\mathrm{r\,MIN}}$, and a greatest element referred to as "maximum reached gain" and denoted by $G_{\mathrm{r\,MAX}}$; and
(b) if the MAA is reciprocal and $\mathbf{Z}_{\mathrm{G}}$ is symmetric, then

$$A_{\mathrm{r}} = \frac{\lambda^2}{4\pi} G_{\mathrm{r\,MAX}}\,. \tag{111}$$

If $N = 1$, that is, if the MAA is a single-port antenna, G_r does not depend on an arbitrary nonzero $\mathbf{I}_{A1}$. Furthermore, if $N = 1$ and the MAA is a reciprocal single-port antenna, it follows from (111) that we have

$$A_r = \frac{\lambda^2}{4\pi} G_r \,. \tag{112}$$

For any $N \geqslant 1$, in a context where the MAA need not be reciprocal and/or $\mathbf{Z}_G$ need not be symmetric, we can use (6), (11) and (68) to obtain

$$G_r = \frac{\mathbf{I}_{A1}^* \mathbf{N}_A \, \mathbf{I}_{A1}}{\mathbf{I}_{A1}^* \, \mathbf{Z}_{AAVGO} \, \mathbf{I}_{A1}} \,. \tag{113}$$

Here, since $\mathbf{Z}_{AAVGO}$ is positive definite, G_r is written as a generalized Rayleigh ratio of $\mathbf{N}_A$ to $\mathbf{Z}_{AAVGO}$, in the variable $\mathbf{I}_{A1}$. It follows that we can use the theorem of Section II to obtain $G_{r\,MAX}$ and $G_{r\,MIN}$. Accordingly, the eigenvalues of $\mathbf{N}_A \mathbf{Z}_{AAVGO}^{-1}$ are real and nonnegative, $G_{r\,MAX}$ is the largest of these eigenvalues, and $G_{r\,MIN}$ is the least of these eigenvalues.

It follows from (69) and (113) that the theorem of Section II leads us to

$$(N > 2) \Longrightarrow (G_{r\,MIN} = 0) \,. \tag{114}$$

IX. MINIMUM REACHED EQUIVALENT AREA AND MEAN REACHED EQUIVALENT AREA

We have discussed the partial reached effective area of the MAA, in Section VII, using the case $n = 1$, and we have discussed the reached effective area of the MAA, in Section VIII, using the case $n = 2$. Let us now study the partial reached effective area of the MAA, in the case $n = 2$, for an arbitrary polarization vector $\mathbf{u}_{pol}$. In Section VI, we have seen that there exists a value $\mathbf{I}_{B2\,pol}$ of $\mathbf{I}_{B2}$ that produces an incident field for which the polarization vector is $\mathbf{u}_{pol}$, that is to say, such that (71) is satisfied.

Thus, if $\mathbf{I}_{B2} = \mathbf{I}_{B2\,pol}$, we have

$$A_{pr}(\mathbf{u}_{pol}) = \eta \frac{P_{BDP1}}{\underset{\sim}{\mathbf{E}}_{B0}^* \underset{\sim}{\mathbf{E}}_{B0}} \tag{115}$$

in the direction $\theta = \theta_R$ and $\varphi = \varphi_R$.

For a practical computation of $A_{pr}(\mathbf{u}_{pol})$, we can use the basis $(\mathbf{u}_{rR}, \mathbf{u}_{\theta R}, \mathbf{u}_{\varphi R})$, (16)–(17) and (32) in (115) to obtain

$$A_{pr}(\mathbf{u}_{pol}) = \eta \frac{\mathbf{I}_{B2\,pol}^* \mathbf{P}^* \overline{\mathbf{h}_B} \, \mathbf{Y}_{AG}^* H(\mathbf{Z}_G) \mathbf{Y}_{AG} \mathbf{h}_B^T \, \mathbf{P} \mathbf{I}_{B2\,pol}}{\mathbf{I}_{B2\,pol}^* \mathbf{I}_{B2\,pol}} \,. \tag{116}$$

It follows that, in this basis,

$$A_{pr}(\mathbf{u}_{pol}) = r_r(\mathbf{I}_{B2\,pol}) \,, \tag{117}$$

where r_r is the Rayleigh ratio defined by (106).

We define the minimum reached equivalent area of the MAA, in a given direction, as the ratio of the average power delivered by the N ports of the MAA used for reception to the power flux density of a uniform plane wave incident on the MAA from the given direction, the polarization of the uniform plane wave being such that it minimizes this average power.

Thus, the minimum reached equivalent area of the MAA in the direction $\theta = \theta_R$ and $\varphi = \varphi_R$, denoted by $A_{req\,MIN}$, is the minimum value of the Rayleigh ratio r_r, which is the least eigenvalue of $\mathbf{N}_{Br}$ according to Rayleigh's theorem.

It follows from (108) that at most $\min\{N, 2\}$ eigenvalues of $\mathbf{N}_{Br}$ are nonzero. The sum of these eigenvalues being $\operatorname{tr} \mathbf{N}_{Br}$, we define the mean reached equivalent area of the MAA, in a given direction, as

$$A_{req\,MEA} = \frac{\operatorname{tr} \mathbf{N}_{Br}}{2} = \frac{A_r + A_{req\,MIN}}{2} \,, \tag{118}$$

because it follows from (32), (71) and (117) that this quantity is a mean value of $A_{pr}(\mathbf{u}_{pol})$ over two orthogonal polarization vectors.

It follows from (108) that

$$(N = 1) \Rightarrow \left(A_{req\,MEA} = \frac{A_r}{2} \text{ and } A_{req\,MIN} = 0 \right). \tag{119}$$

Using the basis $(\mathbf{u}_{rR}, \mathbf{u}_{\theta R}, \mathbf{u}_{\varphi R})$, (32) in (115) and the assumption that $n = 2$ and $\mathbf{I}_{B2} = \mathbf{I}_{B2\,pol}$, we get

$$A_{pr}(\mathbf{u}_{pol}) = \frac{4\lambda^2 r_R^2}{\eta \, |h_R|^2} \frac{P_{BDP1}}{\mathbf{I}_{B2\,pol}^* \mathbf{I}_{B2\,pol}} \,, \tag{120}$$

and using (29) leads us to

$$A_{pr}(\mathbf{u}_{pol}) = \frac{4\lambda^2 r_R^2 \operatorname{Re}(Z_R)}{\eta \, |h_R|^2} \, g_{BT2\,pol} \,, \tag{121}$$

where $g_{BT2\,pol}$ is the unnamed power gain P_{BDP1}/P_{BAVG} in CB in the case $n = 2$, if $\mathbf{I}_{B2} = \mathbf{I}_{B2\,pol}$.

We noted in Section VIII that (101) provides a connection between G_r and g_{AT2}, which are both determined for a specified nonzero excitation. We now observe that (121) provides a connection between $A_{pr}(\mathbf{u}_{pol})$ and $g_{BT2\,pol}$, which is determined for an excitation that is given by (72) as a function of the arbitrary polarization vector $\mathbf{u}_{pol}$. This will now allow us to obtain two results relating to G_r and $A_{pr}(\mathbf{u}_{pol})$, without computation.

By (101) and (121), G_r/g_{AT2} and $A_{pr}(\mathbf{u}_{pol})/g_{BT2\,pol}$ are positive real numbers. It follows that we can use Theorem 15 of [6], (113) and (117) to assert that, in a given direction:

(a) if $\mathbf{M}$ is given by the theorem of Section II applied to the generalized Rayleigh ratio of $\mathbf{N}_A$ to $\mathbf{Z}_{AAVGO}$, in the variable $\mathbf{I}_{A1}$, a mean value of G_r over a number N of linearly independent $\mathbf{I}_{A1}$ is referred to as "mean reached gain" and given by

$$G_{r\,MEA} = \frac{\operatorname{tr} \mathbf{M}}{N} = \frac{\operatorname{tr}\left(\mathbf{N}_A \mathbf{Z}_{AAVGO}^{-1}\right)}{N} \,; \tag{122}$$

and

(b) if the MAA is reciprocal and $\mathbf{Z}_G$ is symmetric, then

$$A_{req\,MEA} = \frac{N\lambda^2}{8\pi} G_{r\,MEA} \tag{123}$$

and

$$(N = 2) \Longrightarrow \left(A_{req\,MIN} = \frac{\lambda^2}{4\pi} G_{r\,MIN} \right). \tag{124}$$

X. INVARIANCE OF THE PARAMETERS

We have frequently utilized the system described in Section III, comprising the MAA and the RAA. This system is now referred to as $\mathcal{S}$. It has two configurations, CA and CB, and two cases corresponding to $n = 1$ and $n = 2$. To define $\mathcal{S}$, we used the initial right-handed rectangular cartesian coordinate system (x, y, z) having its origin O close to the MAA, the associated spherical coordinates system being (r, θ, φ), and the local orthonormal basis of the spherical coordinate system being denoted by $(\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\varphi)$. We also used a point R of coordinates $(r_\mathrm{R}, \theta_\mathrm{R}, \varphi_\mathrm{R})$ in the coordinate system (r, θ, φ), located close to the RAA and such that $\mathrm{OR} = r_\mathrm{R}$ is much larger than the wavelength and the largest dimension of the MAA. In $\mathcal{S}$, we use $\mathcal{B}_\mathrm{R} = (\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$ to denote the local orthonormal basis $(\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\varphi)$ at point R.

We now want to look at what would have happened to the parameters listed in Table 1 to Table 3 if, using the same MAA at the same location, the same origin O and the same point R, we had considered a new system $\mathcal{S}_\mathrm{N}$ defined like $\mathcal{S}$, except that $\mathcal{S}_\mathrm{N}$ would have been built using another right-handed rectangular cartesian coordinate system of $\mathbb{E}$, denoted by $(x_\mathrm{N}, y_\mathrm{N}, z_\mathrm{N})$ and of origin O, in the place of the initial coordinate system (x, y, z) of $\mathbb{E}$ used to build $\mathcal{S}$.

In the case $n = 1$, the parameters listed in Table 1 to Table 3 depend on the vector $\mathbf{u}_\mathrm{pol}$, and we find in Section IV and Section VII that this vector sufficiently determines the vector $\mathbf{h}_\mathrm{R1}$, which characterizes the RAA through (34). Thus, in any direction and for any polarization vector $\mathbf{u}_\mathrm{pol}$ compatible with this direction, the parameters $G_{\mathrm{pa\,MAX}}(\mathbf{u}_\mathrm{pol})$, $G_{\mathrm{pa\,MEA}}(\mathbf{u}_\mathrm{pol})$, $G_{\mathrm{pr\,MAX}}(\mathbf{u}_\mathrm{pol})$, $G_{\mathrm{pr\,MEA}}(\mathbf{u}_\mathrm{pol})$, $A_\mathrm{pa}(\mathbf{u}_\mathrm{pol})$ and $A_\mathrm{pr}(\mathbf{u}_\mathrm{pol})$ are invariant under the change from $\mathcal{S}$ to $\mathcal{S}_\mathrm{N}$. Likewise, for any given nonzero excitation, in any direction and for any polarization vector $\mathbf{u}_\mathrm{pol}$ compatible with this direction, the parameters $G_\mathrm{pa}(\mathbf{u}_\mathrm{pol})$ and $G_\mathrm{pr}(\mathbf{u}_\mathrm{pol})$ are invariant under the change from $\mathcal{S}$ to $\mathcal{S}_\mathrm{N}$.

The case $n = 2$ is more subtle, because the two antennas constituting the RAA in $\mathcal{S}$ are such that $\mathbf{h}_\mathrm{R1} = h_\mathrm{R}\, \mathbf{u}_{\theta\mathrm{R}}$ and $\mathbf{h}_\mathrm{R2} = h_\mathrm{R}\, \mathbf{u}_{\varphi\mathrm{R}}$, where h_R is a nonzero complex number. It follows that the RAA of $\mathcal{S}$ need not be the same as the RAA of $\mathcal{S}_\mathrm{N}$, so that the change from $\mathcal{S}$ to $\mathcal{S}_\mathrm{N}$ is not a mere change of coordinates.

We observe that the definitions of $\mathbf{h}_\mathrm{A}$ in Section III.B and $\mathbf{h}_\mathrm{B}$ in Section III.C are directly applicable to $\mathcal{S}_\mathrm{N}$, and to any orthonormal basis of $\mathbb{E}$. It follows from this fact, and from an adaptation of (67)–(68) and (113) to $\mathcal{S}_\mathrm{N}$, that G_a and G_r are invariant under the change from $\mathcal{S}$ to $\mathcal{S}_\mathrm{N}$. Thus, $G_{\mathrm{a\,MAX}}$, $G_{\mathrm{a\,MEA}}$, $G_{\mathrm{a\,MIN}}$, $G_{\mathrm{r\,MAX}}$, $G_{\mathrm{r\,MEA}}$ and $G_{\mathrm{r\,MIN}}$ are also invariant under the change from $\mathcal{S}$ to $\mathcal{S}_\mathrm{N}$.

To study the behavior of the remaining parameters of Table 3, we will use said cartesian coordinate system $(x_\mathrm{N}, y_\mathrm{N}, z_\mathrm{N})$ of $\mathbb{E}$, the associated spherical coordinates system, denoted by $(r_\mathrm{N}, \theta_\mathrm{N}, \varphi_\mathrm{N})$, and the local orthonormal basis of this spherical coordinate system, denoted by $(\mathbf{u}_{r\mathrm{N}}, \mathbf{u}_{\theta\mathrm{N}}, \mathbf{u}_{\varphi\mathrm{N}})$. We use $(r_\mathrm{NR}, \theta_\mathrm{NR}, \varphi_\mathrm{NR})$ to denote the coordinates of R in the coordinate system $(r_\mathrm{N}, \theta_\mathrm{N}, \varphi_\mathrm{N})$, and we use $\mathcal{B}_\mathrm{NR} = (\mathbf{u}_{r\mathrm{NR}}, \mathbf{u}_{\theta\mathrm{NR}}, \mathbf{u}_{\varphi\mathrm{NR}})$ to denote $(\mathbf{u}_{r\mathrm{N}}, \mathbf{u}_{\theta\mathrm{N}}, \mathbf{u}_{\varphi\mathrm{N}})$ at R. We observe that $r_\mathrm{NR} = r_\mathrm{R}$ and $\mathbf{u}_{r\mathrm{NR}} = \mathbf{u}_{r\mathrm{R}}$.

Since $\mathcal{B}_\mathrm{R}$ and $\mathcal{B}_\mathrm{NR}$ are each a right-handed orthonormal basis of $\mathbb{E}$, and since $\mathbf{u}_{r\mathrm{NR}} = \mathbf{u}_{r\mathrm{R}}$, there exists $\xi \in [0, 2\pi)$ such that the change of basis matrix from $\mathcal{B}_\mathrm{R}$ to $\mathcal{B}_\mathrm{NR}$, denoted by $\mathbf{K}$, is given by

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{pmatrix}. \tag{125}$$

We now use $\mathbf{h}_\mathrm{BR}$ to denote $\mathbf{h}_\mathrm{B}$ determined in $\mathcal{B}_\mathrm{R}$, and $\mathbf{h}_\mathrm{BNR}$ to denote $\mathbf{h}_\mathrm{B}$ determined in $\mathcal{B}_\mathrm{NR}$. Let $\underset{\sim}{\mathbf{E}}_\mathrm{B0}$ be the column vector of the coordinate in $\mathcal{B}_\mathrm{NR}$ of the incident electric field, at the origin O, of an arbitrary uniform plane wave incident from the specified direction. By (14), we have

$$\mathbf{h}_\mathrm{BNR}^\mathrm{T}\, \underset{\sim}{\mathbf{E}}_\mathrm{B0} = \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{K} \underset{\sim}{\mathbf{E}}_\mathrm{B0}\,, \tag{126}$$

where $\underset{\sim}{\mathbf{E}}_\mathrm{B0}$ is an arbitrary vector of the linear span of $\{\mathbf{u}_{\theta\mathrm{NR}}, \mathbf{u}_{\varphi\mathrm{NR}}\}$. All entries of the first row of $\mathbf{h}_\mathrm{BR}$ and $\mathbf{h}_\mathrm{BNR}$ being zeros, it follows from (126) that, for any $\mathbf{X} \in \mathbb{C}^3$ we have $\mathbf{h}_\mathrm{BNR}^\mathrm{T}\mathbf{X} = \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{K}\mathbf{X}$, so that

$$\mathbf{h}_\mathrm{BNR}^\mathrm{T} = \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{K}\,. \tag{127}$$

We have

$$\begin{aligned} \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{K}\mathbf{P} = \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{K} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} &= \mathbf{h}_\mathrm{BR}^\mathrm{T} \begin{pmatrix} 0 & 0 \\ \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix} \\ &= \mathbf{h}_\mathrm{BR}^\mathrm{T}\mathbf{P} \begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}. \end{aligned} \tag{128}$$

It follows from (127)–(128) that

$$\mathbf{h}_\mathrm{BNR}^\mathrm{T}\mathbf{P} = \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{P}\mathbf{L}\,, \tag{129}$$

where $\mathbf{L}$ is the orthogonal matrix

$$\mathbf{L} = \begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}. \tag{130}$$

According to Section V and Section VI, A_a, $A_{\mathrm{aeq\,MIN}}$ and $A_{\mathrm{aeq\,MEA}}$ can be computed, in $\mathcal{S}$, as the largest eigenvalue, the least eigenvalue and the half trace of the positive semidefinite matrix $\mathbf{N}_\mathrm{Ba}$ given by

$$\mathbf{N}_\mathrm{Ba} = \frac{\eta}{4}\, \mathbf{P}^\mathrm{T}\, \overline{\mathbf{h}_\mathrm{BR}}\, H(\mathbf{Z}_\mathrm{A})^{-1} \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{P}\,. \tag{131}$$

It follows from the analysis of Section V and Section VI adapted to the system $\mathcal{S}_\mathrm{N}$, that A_a, $A_{\mathrm{aeq\,MIN}}$ and $A_{\mathrm{aeq\,MEA}}$ can be computed, in $\mathcal{S}_\mathrm{N}$, as the largest eigenvalue, the least eigenvalue and the half trace of a positive semidefinite matrix $\mathbf{N}_\mathrm{NBa}$ given by

$$\mathbf{N}_\mathrm{NBa} = \frac{\eta}{4}\, \mathbf{P}^\mathrm{T}\, \overline{\mathbf{h}_\mathrm{BNR}}\, H(\mathbf{Z}_\mathrm{A})^{-1} \mathbf{h}_\mathrm{BNR}^\mathrm{T}\, \mathbf{P}\,. \tag{132}$$

Using (129) in (132), we get

$$\mathbf{N}_\mathrm{NBa} = \frac{\eta}{4}\, \mathbf{L}^{-1}\mathbf{P}^\mathrm{T}\, \overline{\mathbf{h}_\mathrm{BR}}\, H(\mathbf{Z}_\mathrm{A})^{-1} \mathbf{h}_\mathrm{BR}^\mathrm{T}\, \mathbf{P}\mathbf{L}\,. \tag{133}$$

A comparison of (131) and (133) shows that $\mathbf{N}_\mathrm{NBa}$ and $\mathbf{N}_\mathrm{Ba}$ are similar, and consequently have the same eigenvalues and the same trace. Thus, A_a, $A_{\mathrm{aeq\,MIN}}$ and $A_{\mathrm{aeq\,MEA}}$ are invariant under the change from $\mathcal{S}$ to $\mathcal{S}_\mathrm{N}$.

According to Section VIII and Section IX, $A_{\rm r}$, $A_{\rm req\,MIN}$ and $A_{\rm req\,MEA}$ can be computed, in $\mathcal{S}$, as the largest eigenvalue, the least eigenvalue and the half trace of the positive semidefinite matrix $\mathbf{N}_{\rm Br}$ given by

$$\mathbf{N}_{\rm Br} = \eta\, \mathbf{P}^{\rm T}\, \overline{\mathbf{h}_{\rm BR}}\, \mathbf{Y}_{\rm AG}^{*} H(\mathbf{Z}_{\rm G}) \mathbf{Y}_{\rm AG} \mathbf{h}_{\rm BR}^{\rm T}\, \mathbf{P}\,. \tag{134}$$

It follows from the analysis of Section VIII and Section IX adapted to the system $\mathcal{S}_{\rm N}$, that $A_{\rm r}$, $A_{\rm req\,MIN}$ and $A_{\rm req\,MEA}$ can be computed, in $\mathcal{S}_{\rm N}$, as the largest eigenvalue, the least eigenvalue and the half trace of a positive semidefinite matrix $\mathbf{N}_{\rm NBr}$ given by

$$\mathbf{N}_{\rm NBr} = \eta\, \mathbf{P}^{\rm T}\, \overline{\mathbf{h}_{\rm BNR}}\, \mathbf{Y}_{\rm AG}^{*} H(\mathbf{Z}_{\rm G}) \mathbf{Y}_{\rm AG} \mathbf{h}_{\rm BNR}^{\rm T}\, \mathbf{P}\,. \tag{135}$$

Using (129) in (135), we get

$$\mathbf{N}_{\rm NBr} = \eta\, \mathbf{L}^{-1}\mathbf{P}^{\rm T}\, \overline{\mathbf{h}_{\rm BR}}\, \mathbf{Y}_{\rm AG}^{*} H(\mathbf{Z}_{\rm G}) \mathbf{Y}_{\rm AG} \mathbf{h}_{\rm BR}^{\rm T}\, \mathbf{P}\mathbf{L}\,. \tag{136}$$

A comparison of (134) and (136) shows that $\mathbf{N}_{\rm NBr}$ and $\mathbf{N}_{\rm Br}$ are similar, and consequently have the same eigenvalues and the same trace. Thus, $A_{\rm r}$, $A_{\rm req\,MIN}$ and $A_{\rm req\,MEA}$ are invariant under the change from $\mathcal{S}$ to $\mathcal{S}_{\rm N}$.

XI. RANDOM POLARIZATION

A. SURFACE ELEMENT OF $\mathbb{S}_2$

Before considering the probability density of a random polarization, we need to study the surface element of $\mathbb{S}_2$.

A simple parametrization of $\mathbb{S}_2$ uses 3 real parameters χ_1, ϕ_0 and ϕ_1, in such a way that an arbitrary element of $\mathbb{S}_2$ is given by

$$\hat{\mathbf{x}} = \begin{pmatrix} \sin\chi_1\ e^{j\phi_1} \\ \cos\chi_1\ e^{j\phi_0} \end{pmatrix}, \tag{137}$$

where $\chi_1 \in [0, \pi/2]$, $\phi_0 \in [0, 2\pi)$ and $\phi_1 \in [0, 2\pi)$. To define a surface element and a surface area of $\mathbb{S}_2$, we identify $\mathbb{S}_2$ with the unit hypersphere of $\mathbb{R}^4$, using an isometric isomorphism $\Psi_2 : \mathbb{C}^2 \to \mathbb{R}^4$, where $\mathbb{C}^2$ is regarded as a normed vector space over $\mathbb{R}$, such that a parametric equation of $\Psi_2(\mathbb{S}_2)$ is

$$\breve{\mathbf{x}} = \begin{pmatrix} \sin\chi_1 \cos\phi_1 \\ \sin\chi_1 \sin\phi_1 \\ \cos\chi_1 \cos\phi_0 \\ \cos\chi_1 \sin\phi_0 \end{pmatrix}, \tag{138}$$

where $\breve{\mathbf{x}}$ denotes an arbitrary element of the unit hypersphere of $\mathbb{R}^4$. It follows that the surface element of $\mathbb{S}_2$ is

$$dS_2 = \cos\chi_1 \sin\chi_1\ d\chi_1\, d\phi_0\, d\phi_1\,, \tag{139}$$

which leads us to the surface area of $\mathbb{S}_2$:

$$S_2 = \iiint\limits_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} dS_2 = 2\pi^2\,. \tag{140}$$

Let ν be a positive integer. Some authors use S_ν, and others $S_{\nu+1}$, to denote the surface area of a hypersphere of $\mathbb{R}^{\nu+1}$, of unit radius. According to our notations, S_2 is the surface area of $\mathbb{S}_2$, which is equal to the surface area of the unit hypersphere of $\mathbb{R}^4$. Consequently, (140) is a well known result [16, p. 877].

B. PROBABILITY DENSITY

In CB, in the direction $\theta = \theta_{\rm R}$ and $\varphi = \varphi_{\rm R}$, and in the case $n = 2$, we saw in Section VI that, for an arbitrary polarization vector $\mathbf{u}_{\rm pol}$, there exists a value $\mathbf{I}_{\rm B2\,pol}$ of $\mathbf{I}_{\rm B2}$ that produces an incident field for which the polarization vector is $\mathbf{u}_{\rm pol}$.

Accordingly, to define a uniform plane wave incident from the direction $\theta = \theta_{\rm R}$ and $\varphi = \varphi_{\rm R}$, the plane wave having a random polarization, we only need to define a random nonzero $\mathbf{I}_{\rm B2} \in \mathbb{C}^2$. In what follows, the random nonzero $\mathbf{I}_{\rm B2} \in \mathbb{C}^2$ is denoted by $\mathbf{x}$.

Let $\mathbf{N}$ be a hermitian matrix of size 2 by 2. Let r be the Rayleigh ratio

$$r(\mathbf{x}) = \frac{\mathbf{x}^*\mathbf{N}\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\,. \tag{141}$$

As regards $r(\mathbf{x})$, the random nonzero $\mathbf{x} \in \mathbb{C}^2$ is fully characterized by the probability density of $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2$, which lies in $\mathbb{S}_2$, because $r(\mathbf{x}) = r(\hat{\mathbf{x}})$. We therefore introduce the probability density $f : \mathbb{S}_2 \to \mathbb{R}$ of $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}||_2$. Using (32) and (71), we find that, in the basis $(\mathbf{u}_{r\rm R}, \mathbf{u}_{\theta\rm R}, \mathbf{u}_{\varphi\rm R})$,

$$\underset{\sim}{\mathbf{u}}_{\rm pol} = \frac{\underset{\sim}{\mathbf{E}}_{\rm B0}}{\sqrt{\underset{\sim}{\mathbf{E}}_{\rm B0}^{*}\underset{\sim}{\mathbf{E}}_{\rm B0}}} = \mathbf{P}\,\hat{\mathbf{x}}\,. \tag{142}$$

Thus, f fully characterizes the random $\mathbf{u}_{\rm pol}$, and may therefore be referred to as the "probability density of the random polarization". It must satisfy

$$\iiint\limits_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} f(\hat{\mathbf{x}})\, dS_2 = 1\,. \tag{143}$$

The probability density of the random polarization may for instance be a uniform probability density $f_{\rm U} : \mathbb{S}_2 \to \mathbb{R}$, which does not depend on χ_1, ϕ_0 and ϕ_1. In this case, it follows from (140) and (143) that, for any $\hat{\mathbf{x}} \in \mathbb{S}^2$,

$$f_{\rm U}(\hat{\mathbf{x}}) = \frac{1}{S_2} = \frac{1}{2\pi^2}\,. \tag{144}$$

C. EXPECTED VALUE OF THE RAYLEIGH RATIO

The expected value of the Rayleigh ratio r for the probability density f is

$$\langle r(\hat{\mathbf{x}})\rangle_f = \iiint\limits_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} f(\hat{\mathbf{x}}) r(\hat{\mathbf{x}})\, dS_2\,. \tag{145}$$

Thus,

$$\langle r(\hat{\mathbf{x}})\rangle_f = \iiint\limits_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} f(\hat{\mathbf{x}})\ \hat{\mathbf{x}}^*\mathbf{N}\hat{\mathbf{x}}\ dS_2\,. \tag{146}$$

$\mathbf{N}$ being hermitian, let λ_1 and λ_2 be the real eigenvalues of $\mathbf{N}$, counting multiplicity, and let $(\mathbf{u}_1, \mathbf{u}_2)$ be an orthonormal basis of $\mathbb{C}^2$ such that, for any $p \in \{1, 2\}$, $\mathbf{u}_p$ is an eigenvector of $\mathbf{N}$ associated with the eigenvalue λ_p. We obtain

$$\langle r(\hat{\mathbf{x}})\rangle_f = \iiint\limits_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} f(\hat{\mathbf{x}}) \sum_{p=1}^{2} \lambda_p \left|\langle\hat{\mathbf{x}}, \mathbf{u}_p\rangle\right|^2 dS_2\,, \tag{147}$$

where, for any $\mathbf{a}$ and $\mathbf{b}$ lying in $\mathbb{C}^2$, $\langle\mathbf{a}, \mathbf{b}\rangle = \mathbf{b}^*\mathbf{a}$, in line with the notations of Appendix A. This leads us to

$$\langle r(\hat{\mathbf{x}})\rangle_f = \sum_{p=1}^{2} \lambda_p \iiint\limits_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} f(\hat{\mathbf{x}}) \left|\langle\hat{\mathbf{x}}, \mathbf{u}_p\rangle\right|^2 dS_2\,. \tag{148}$$

We now assume that f is the uniform probability density f_{U}. It follows from (144) and (148) that

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\mathrm{U}}} = \frac{1}{2\pi^2}\sum_{p=1}^{2}\lambda_p \iiint_{\hat{\mathbf{x}}\,\in\,\mathbb{S}_2} |\langle\hat{\mathbf{x}},\mathbf{u}_p\rangle|^2 \, dS_2 \,. \tag{149}$$

At this point, we need to compute the integral

$$I = \iiint_{\hat{\mathbf{x}}\in\mathbb{S}_2} |\langle\hat{\mathbf{x}},\mathbf{u}\rangle|^2 \, dS_2 \,. \tag{150}$$

for an arbitrary fixed $\mathbf{u} \in \mathbb{S}_2$. Without loss of generality, we can assume that $\mathbf{u} = (0,1)^{\mathrm{T}}$. Using (137) and (139), we get

$$I = \iiint_{\hat{\mathbf{x}}\in\mathbb{S}_2} |\cos\chi_1\, e^{j\phi_0}|^2 \, \cos\chi_1 \sin\chi_1 \, d\chi_1\, d\phi_0\, d\phi_1 \,, \tag{151}$$

and then

$$I = \iiint_{\hat{\mathbf{x}}\in\mathbb{S}_2} \cos^3\chi_1 \sin\chi_1 \, d\chi_1\, d\phi_0\, d\phi_1 \,, \tag{152}$$

so that

$$I = (2\pi)^2 \int_0^{\pi/2} \cos^3\chi_1 \sin\chi_1 \, d\chi_1 = \pi^2 \,. \tag{153}$$

It follows from (149) and (153) that the expected value of the Rayleigh ratio r for the probability density f_{U} is

$$\langle r(\hat{\mathbf{x}})\rangle_{f_{\mathrm{U}}} = \frac{1}{2}\sum_{p=1}^{2}\lambda_p = \frac{\operatorname{tr}\mathbf{N}}{2} \,. \tag{154}$$

D. MEAN ABSOLUTE EQUIVALENT AREA

By (75), (76) and (154), we have

$$A_{\mathrm{aeq\,MEA}} = \langle A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})\rangle_{f_{\mathrm{U}}} \,. \tag{155}$$

This result may be phrased by saying that the mean absolute equivalent area of the MAA in a given direction is the expected value of the partial absolute effective area of the MAA in this direction, for a random polarization having a uniform probability density.

To establish this result, we have used the system $\mathcal{S}$, but we can employ the results of Section X relating to the invariance of $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ and $A_{\mathrm{aeq\,MEA}}$ to assert that (155) is invariant under the change from $\mathcal{S}$ to $\mathcal{S}_{\mathrm{N}}$.

E. MEAN REACHED EQUIVALENT AREA

By (117), (118) and (154), we have

$$A_{\mathrm{req\,MEA}} = \langle A_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})\rangle_{f_{\mathrm{U}}} \,. \tag{156}$$

This result may be phrased by saying that the mean reached equivalent area of the MAA in a given direction is the expected value of the partial reached effective area of the MAA in this direction, for a random polarization having a uniform probability density.

To establish this result, we have used the system $\mathcal{S}$, but we can employ the results of Section X relating to the invariance of $A_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})$ and $A_{\mathrm{req\,MEA}}$ to assert that (156) is invariant under the change from $\mathcal{S}$ to $\mathcal{S}_{\mathrm{N}}$.

XII. GENERALIZATION, COMMENTS AND EXAMPLE

A. GENERALIZATION OF “ABSOLUTE” PARAMETERS

It follows from their definitions using the concept of available power that A_{a}, $A_{\mathrm{aeq\,MEA}}$ and $A_{\mathrm{aeq\,MIN}}$ do not depend on what is connected to the MAA during reception. For the same reason, $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ does not depend on what is connected to the MAA during reception, for any polarization vector $\mathbf{u}_{\mathrm{pol}}$. This is why $\mathbf{Y}_{\mathrm{G}}$ and $\mathbf{Z}_{\mathrm{G}}$ do not appear in the formulas (44) and (60)–(61) relating to the computation of A_{a}, $A_{\mathrm{aeq\,MEA}}$, $A_{\mathrm{aeq\,MIN}}$ and $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$. This also means that all our assumptions regarding the MLA are irrelevant to the definitions and the values of A_{a}, $A_{\mathrm{aeq\,MEA}}$, $A_{\mathrm{aeq\,MIN}}$ and $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$.

Since we have used $\mathbf{I}_{\mathrm{A1}}$ as the variable defining the excitation when the MAA is used for emission, G_{a} and $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$ depend on $\mathbf{I}_{\mathrm{A1}}$ but not otherwise on the MGA. Thus, $G_{\mathrm{a\,MAX}}$, $G_{\mathrm{a\,MEA}}$ and $G_{\mathrm{a\,MIN}}$ do not depend on the MGA, and $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ and $G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}})$ do not depend on the MGA for any polarization vector $\mathbf{u}_{\mathrm{pol}}$. This is why $\mathbf{Y}_{\mathrm{G}}$ and $\mathbf{Z}_{\mathrm{G}}$ do not appear in the formulas (47)–(48) and (67)–(68) relating to the computation of $G_{\mathrm{a\,MAX}}$, $G_{\mathrm{a\,MEA}}$, $G_{\mathrm{a\,MIN}}$, $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ and $G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}})$. This also means that all our assumptions regarding the MGA, except its ability to cause the wanted $\mathbf{I}_{\mathrm{A1}}$, are irrelevant to the definitions and the values of G_{a}, $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, $G_{\mathrm{a\,MAX}}$, $G_{\mathrm{a\,MEA}}$, $G_{\mathrm{a\,MIN}}$, $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ and $G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}})$.

Thus, the definitions of the 11 parameters of Table 1 to Table 3 that include “absolute” in their names do not depend on the characteristics of the device connected to the MAA, provided that, in the case of the 7 parameters concerning emission, this device can cause the assumed excitation.

B. MAXIMUM AND MINIMUM EMISSION PARAMETERS

The 4 emission parameters $G_{\mathrm{a\,MAX}}$, $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$, $G_{\mathrm{r\,MAX}}$ and $G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ that include “maximum” in their names have been defined as the greatest elements of the sets of the values of G_{a}, $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, G_{r} and $G_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})$, respectively, obtained for all nonzero $\mathbf{I}_{\mathrm{A1}}$, or equivalently for all nonzero $\mathbf{V}_{\mathrm{AO1}}$. However, it follows from a basic property mentioned in Section II that $G_{\mathrm{a\,MAX}}$, $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$, $G_{\mathrm{r\,MAX}}$ and $G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})$ are also the greatest elements of the sets of the values of G_{a}, $G_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})$, G_{r} and $G_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})$, respectively, obtained for all $\mathbf{I}_{\mathrm{A1}} \in \mathbb{S}_N$, or equivalently for all $\mathbf{V}_{\mathrm{AO1}} \in \mathbb{S}_N$.

The 2 emission parameters $G_{\mathrm{a\,MIN}}$ and $G_{\mathrm{r\,MIN}}$ that include “minimum” in their names have been defined as the least elements of the sets of the values of G_{a} and G_{r}, respectively, obtained for all nonzero $\mathbf{I}_{\mathrm{A1}}$, or equivalently for all nonzero $\mathbf{V}_{\mathrm{AO1}}$. However, $G_{\mathrm{a\,MIN}}$ and $G_{\mathrm{r\,MIN}}$ are also the least elements of the sets of the values of G_{a} and G_{r}, respectively, obtained for all $\mathbf{I}_{\mathrm{A1}} \in \mathbb{S}_N$, or equivalently for all $\mathbf{V}_{\mathrm{AO1}} \in \mathbb{S}_N$.

C. EXAMPLE

We consider an arrangement of three perfectly conducting parallel cylindrical center-fed dipole antennas lying in free space, forming an MAA. The total length of each antenna is $\ell = 0.94\,\lambda/2$, and their spacing is $D = \lambda/2$. All antennas have the same wire diameter $\ell/50$. The coordinates of the end points of Antenna 1 are $(-D, -\ell\sqrt{2}/4, -\ell\sqrt{2}/4)$ and $(-D, \ell\sqrt{2}/4, \ell\sqrt{2}/4)$. The coordinates of the end points of Antenna 2 are $(0, -\ell/2, 0)$ and $(0, \ell/2, 0)$. The coordinates of the end points of Antenna 3 are $(D, -\ell\sqrt{2}/4, \ell\sqrt{2}/4)$ and $(D, \ell\sqrt{2}/4, -\ell\sqrt{2}/4)$. Thus, no two of these antennas are parallel.

Using NEC-2D, we can directly compute $\mathbf{Y}_{\mathrm{A}} = \mathbf{Z}_{\mathrm{A}}^{-1}$ and the matrix

$$\mathbf{F}_{\mathrm{A}} = j\eta \frac{e^{-jkr}}{2\lambda} \mathbf{h}_{\mathrm{A}} \mathbf{Y}_{\mathrm{A}} \,, \tag{157}$$

in several directions. The computed values are

$$\mathbf{Y}_{\mathrm{A}} \simeq \begin{pmatrix} 12.5-0.7j & 2.5+2.8j & -0.2+1.1j \\ 2.5+2.8j & 12.3+0.4j & 2.5+2.8j \\ -0.2+1.1j & 2.5+2.8j & 12.5-0.7j \end{pmatrix} \mathrm{mS} \tag{158}$$

and

$$\mathbf{F}_{\mathrm{A}} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0.148+0.401j & -0.209-0.568j & 0.148+0.402j \\ 0.050+0.556j & 0 & -0.050-0.556j \end{pmatrix} \tag{159}$$

in the basis $(\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z)$ and the direction of $\mathbf{u}_x$. This leads us to

$$\mathbf{h}_{\mathrm{A}} = \mathbf{h}_{\mathrm{B}} \simeq \frac{\lambda}{10} \begin{pmatrix} 0 & 0 & 0 \\ -2.31-0.13j & 0 & 2.31+0.13j \\ 2.41+0.07j & -3.41-0.11j & 2.41+0.07j \end{pmatrix} \tag{160}$$

in the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$ and the direction $\theta = \pi/2$ and $\varphi = 0$. In this direction, we define one of the two possible circular polarizations by the vector of coordinates

$$\underset{\sim}{\mathbf{u}}_{\mathrm{pol}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ j \end{pmatrix} \tag{161}$$

in the basis $(\mathbf{u}_{r\mathrm{R}}, \mathbf{u}_{\theta\mathrm{R}}, \mathbf{u}_{\varphi\mathrm{R}})$.

As regards the corresponding partial absolute parameters, we obtain the values $G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}}) \simeq 2.22$, $G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}}) \simeq 0.740$ and $A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) \simeq 0.177\,\lambda^2$, which are compatible with (42).

As regards the other absolute parameters, we get $G_{\mathrm{a\,MAX}} \simeq 2.80$, $G_{\mathrm{a\,MEA}} \simeq 1.48$, $G_{\mathrm{a\,MIN}} \simeq 0.00$, $A_{\mathrm{a}} \simeq 0.223\,\lambda^2$, $A_{\mathrm{aeq\,MEA}} \simeq 0.177\,\lambda^2$ and $A_{\mathrm{aeq\,MIN}} \simeq 0.130\,\lambda^2$, which are compatible with (65), (70) and (81).

Assuming

$$\mathbf{Z}_{\mathrm{G}} = \begin{pmatrix} 75+3j & -2j & 0 \\ -2j & 75+2j & -2j \\ 0 & -2j & 75+3j \end{pmatrix} \Omega \,, \tag{162}$$

which is symmetric and has a positive definite hermitian part, we can investigate the reached parameters.

As regards the partial reached parameters for the circular polarization defined by (161), we obtain the values $G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}}) \simeq 2.10$, $G_{\mathrm{pr\,MEA}}(\mathbf{u}_{\mathrm{pol}}) \simeq 0.700$ and $A_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}}) \simeq 0.167\,\lambda^2$, which are compatible with (91).

As regards the other reached parameters, we get $G_{\mathrm{r\,MAX}} \simeq 2.58$, $G_{\mathrm{r\,MEA}} \simeq 1.40$, $G_{\mathrm{r\,MIN}} \simeq 0.00$, $A_{\mathrm{r}} \simeq 0.205\,\lambda^2$, $A_{\mathrm{req\,MEA}} \simeq 0.167\,\lambda^2$ and $A_{\mathrm{req\,MIN}} \simeq 0.129\,\lambda^2$, which are compatible with (111), (114) and (123).

XIII. RESULTS RELATED TO THE FRIIS FORMULA

Two generalizations of the Friis transmission formula to radio transmission between reciprocal, passive and LTI MAAs have recently been proposed [17]–[19]. They use the concept of unnamed power gain for LTI multiports [6, Sec. IX], and are applicable at any distance, in an arbitrary reciprocal and passive LTI environment. Moreover, the generalization investigated in [17] is based on Theorem 29 of [6] and the analysis of [6, Sec. XIII].

We are going to establish a new generalization of the Friis transmission formula to two passive and LTI MAAs, called MAA 1 and MAA 2, lying in the isotropic, homogeneous and lossless passive medium of Section III.A, the distance between the MAAs, denoted by d, being much larger than the wavelength and their largest dimensions. MAA 1 has N_1 ports numbered from 1 to N_1, and MAA 2 has N_2 ports numbered from 1 to N_2. We assume that MAA 1 has an impedance matrix, denoted by $\mathbf{Z}_{\mathrm{A1}}$ and of size N_1 by N_1, and that $H(\mathbf{Z}_{\mathrm{A1}})$ is positive definite. We assume that MAA 2 has an impedance matrix, denoted by $\mathbf{Z}_{\mathrm{A2}}$ and of size N_2 by N_2, and that $H(\mathbf{Z}_{\mathrm{A2}})$ is positive definite.

For a specified nonzero excitation applied to the ports of MAA 1 used for emission, let G_{AU} be the unnamed power gain between MAA 1 and MAA 2 used for reception, that is to say, the ratio of the available power at the ports of MAA 2 to the average power received by the ports of MAA 1. For a specified nonzero excitation applied to the ports of MAA 2 used for emission, let G_{BU} be the unnamed power gain between MAA 2 and MAA 1 used for reception, that is to say, the ratio of the available power at the ports of MAA 1 to the average power received by the ports of MAA 2. By Theorem 29 of [6], G_{AU} has a maximum value for all nonzero excitations applied to MAA 1, denoted by $G_{\mathrm{AU\,MAX}}$, and G_{BU} has a maximum value for all nonzero excitations applied to MAA 2, denoted by $G_{\mathrm{BU\,MAX}}$.

Let $G_{\mathrm{a\,1\,MAX}}$ be the maximum absolute gain, and $A_{\mathrm{a\,1}}$ the absolute effective area, of MAA 1 in the direction of MAA 2. Let $G_{\mathrm{a\,2\,MAX}}$ be the maximum absolute gain, and $A_{\mathrm{a\,2}}$ the absolute effective area, of MAA 2 in the direction of MAA 1. It follows from the definitions that:

$$G_{\mathrm{AU\,MAX}} \leqslant \frac{G_{\mathrm{a\,1\,MAX}}\, A_{\mathrm{a\,2}}}{4\pi\, d^2} \tag{163}$$

and

$$G_{\mathrm{BU\,MAX}} \leqslant \frac{G_{\mathrm{a\,2\,MAX}}\, A_{\mathrm{a\,1}}}{4\pi\, d^2} \,. \tag{164}$$

Let $G_{\mathrm{a\,1}}$ be the absolute gain for a specified nonzero excitation, and $A_{\mathrm{pa\,1}}(\mathbf{u}_{\mathrm{pol}})$ a partial absolute effective area, of MAA 1 in the direction of MAA 2. Let $G_{\mathrm{a\,2}}$ be the absolute gain for a specified nonzero excitation, and $A_{\mathrm{pa\,2}}(\mathbf{u}_{\mathrm{pol}})$ a partial absolute effective area, of MAA 2 in the direction of MAA 1.

If we assume that there exists an excitation of MAA 1 for which $G_{\mathrm{a\,1}} = G_{\mathrm{a\,1\,MAX}}$ in the direction of MAA 2, this excitation producing, at a large distance in the direction of MAA 2, a uniform incident plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,1}}$ such that $A_{\mathrm{a\,2}} = A_{\mathrm{pa\,2}}(\mathbf{u}_{\mathrm{pol\,1}})$, then:

(a) for this excitation, we have

$$G_{\rm AU} = \frac{G_{\rm a\,1\,MAX}\, A_{\rm a\,2}}{4\pi\, d^2}\,; \tag{165}$$

(b) it follows from (163) and (165) that

$$G_{\rm AU\,MAX} = \frac{G_{\rm a\,1\,MAX}\, A_{\rm a\,2}}{4\pi\, d^2}\,; \tag{166}$$

(c) if MAA 2 is reciprocal, it follows from (65) and (166) that

$$G_{\rm AU\,MAX} = G_{\rm a\,1\,MAX}\, G_{\rm a\,2\,MAX} \left(\frac{\lambda}{4\pi\, d}\right)^2\,; \tag{167}$$

and

(d) if MAA 1 is reciprocal, it follows from (65) and (166) that

$$G_{\rm AU\,MAX} = \frac{A_{\rm a\,1}\, A_{\rm a\,2}}{d^2\,\lambda^2}\,. \tag{168}$$

Likewise, if there exists an excitation of MAA 2 for which $G_{\rm a\,2} = G_{\rm a\,2\,MAX}$ in the direction of MAA 1, this excitation producing, at a large distance in the direction of MAA 1, a uniform incident plane wave of polarization vector $\mathbf{u}_{\rm pol\,2}$ such that $A_{\rm a\,1} = A_{\rm pa\,1}(\mathbf{u}_{\rm pol\,2})$, then:

(e) for this excitation, we have

$$G_{\rm BU} = \frac{G_{\rm a\,2\,MAX}\, A_{\rm a\,1}}{4\pi\, d^2}\,; \tag{169}$$

(f) it follows from (164) and (169) that

$$G_{\rm BU\,MAX} = \frac{G_{\rm a\,2\,MAX}\, A_{\rm a\,1}}{4\pi\, d^2}\,; \tag{170}$$

(g) if MAA 1 is reciprocal, it follows from (65) and (170) that

$$G_{\rm BU\,MAX} = G_{\rm a\,1\,MAX}\, G_{\rm a\,2\,MAX} \left(\frac{\lambda}{4\pi\, d}\right)^2\,; \tag{171}$$

and

(h) if MAA 2 is reciprocal, it follows from (65) and (170) that

$$G_{\rm BU\,MAX} = \frac{A_{\rm a\,1}\, A_{\rm a\,2}}{d^2\,\lambda^2}\,. \tag{172}$$

We observe that we can also assert logical converses of the statements (b) and (f), as follows:

(i) if (166) is satisfied, there exists an excitation of MAA 1 such that (165) is satisfied, this excitation being necessarily such that $G_{\rm a\,1} = G_{\rm a\,1\,MAX}$ in the direction of MAA 2, and such that it produces, at a large distance in the direction of MAA 2, an incident uniform plane wave of polarization vector $\mathbf{u}_{\rm pol\,1}$ such that $A_{\rm a\,2} = A_{\rm pa\,2}(\mathbf{u}_{\rm pol\,1})$; and

(j) if (170) is satisfied, there exists an excitation of MAA 2 such that (169) is satisfied, this excitation being necessarily such that $G_{\rm a\,2} = G_{\rm a\,2\,MAX}$ in the direction of MAA 1, and such that it produces, at a large distance in the direction of MAA 1, an incident uniform plane wave of polarization vector $\mathbf{u}_{\rm pol\,2}$ such that $A_{\rm a\,1} = A_{\rm pa\,1}(\mathbf{u}_{\rm pol\,2})$.

Consequently, if MAA 1 and MAA 2 are reciprocal, we can assert that the statement $\mathcal{S}_1$ = "there exists an excitation of MAA 1 for which $G_{\rm a\,1} = G_{\rm a\,1\,MAX}$ in the direction of MAA 2, this excitation producing, at a large distance in the direction of MAA 2, a uniform incident plane wave of polarization vector $\mathbf{u}_{\rm pol\,1}$ such that $A_{\rm a\,2} = A_{\rm pa\,2}(\mathbf{u}_{\rm pol\,1})$" and the statement $\mathcal{S}_2$ = "there exists an excitation of MAA 2 for which $G_{\rm a\,2} = G_{\rm a\,2\,MAX}$ in the direction of MAA 1, this excitation producing, at a large distance in the direction of MAA 1, a uniform incident plane wave of polarization vector $\mathbf{u}_{\rm pol\,2}$ such that $A_{\rm a\,1} = A_{\rm pa\,1}(\mathbf{u}_{\rm pol\,2})$" are equivalent.

Proof: To show this, we can for instance assume that $\mathcal{S}_1$ is true. Thus, (167) is true because MAA 2 is reciprocal. Since MAA 1, MAA 2 and the medium surrounding them are reciprocal, we can use Theorem 29 of [6] and the explanations of Appendix B to assert that $G_{\rm AU\,MAX} = G_{\rm BU\,MAX}$, so that (167) entails (171). Using (65), we find that (170) is true. Using (j), we find that $\mathcal{S}_2$ is true. □

The transmission formulas (166)–(168) subject to the condition $\mathcal{S}_1$, the transmission formulas (170)–(172) subject to the condition $\mathcal{S}_2$, and the equivalence between $\mathcal{S}_1$ and $\mathcal{S}_2$ if MAA 1 and MAA 2 are reciprocal, form a new generalization of the Friis transmission formula. In the special case where $N_1 = N_2 = 1$, we have $G_{\rm AU} = G_{\rm AU\,MAX}$ for any nonzero excitation of MAA 1, and $G_{\rm BU} = G_{\rm BU\,MAX}$ for any nonzero excitation of MAA 2, so that (166)–(168) and (170)–(172) exactly correspond to 3 traditional forms of the Friis transmission formula for transmission between two antennas [17, Sec. I], [20]–[21], [22, Sec. 4.4.2]. These 3 traditional forms being subject to the condition that the antennas are polarization matched, we see that this condition is replaced with $\mathcal{S}_1$ and $\mathcal{S}_2$ in our new generalization of the Friis formula to transmission between two MAAs. We also note that this generalization of the Friis transmission formula uses parameters that do not depend on the characteristics of the devices connected to the MAAs, as explained in Section XII.A.

XIV. OTHER TRANSMISSION FORMULAS

We are now going to consider other new transmission formulas, which are based on the properties of the transducer power gain stated in Theorem 4 of [5] or Theorem 15 of [6], and use parameters that depend on the characteristics of the devices connected to the MAAs.

We consider the same passive and LTI MAAs as the ones defined in Section XIII, in the same medium and the same configuration. When MAA 1 is used for emission and MAA 2 for reception, MAA 1 is connected to an LTI N_1-port generator of internal impedance matrix $\mathbf{Z}_{S1}$, and MAA 2 is connected to an LTI N_2-port load of impedance matrix $\mathbf{Z}_{S2}$. When MAA 2 is used for emission and MAA 1 for reception, MAA 1 is connected to an LTI N_1-port load of impedance matrix $\mathbf{Z}_{S1}$, and MAA 2 is connected to an LTI N_2-port generator of internal impedance matrix $\mathbf{Z}_{S2}$. We assume that $H(\mathbf{Z}_{S1})$ and $H(\mathbf{Z}_{S2})$ are positive definite.

For a specified nonzero excitation applied to the ports of MAA 1 used for emission, let $G_{\rm AT}$ be the transducer power gain between MAA 1 and MAA 2 used for reception, that is to say, the ratio of the average power delivered by the ports of MAA 2 to the available power at the ports of the LTI N_1-port generator connected to the ports of MAA 1. For a specified nonzero excitation applied to the ports of MAA 2 used for emission, let G_{BT} be the transducer power gain between MAA 2 and MAA 1 used for reception, that is to say, the ratio of the average power delivered by the ports of MAA 1 to the available power at the ports of the LTI N_2-port generator connected to the ports of MAA 2. By Theorem 4 of [5] or Theorem 15 of [6], G_{AT} has a maximum value for all nonzero excitations applied to MAA 1, denoted by $G_{\mathrm{AT\,MAX}}$, and G_{BT} has a maximum value for all nonzero excitations applied to MAA 2, denoted by $G_{\mathrm{BT\,MAX}}$.

Let $G_{\mathrm{r\,1\,MAX}}$ be the maximum reached gain, and $A_{\mathrm{r\,1}}$ the reached effective area, of MAA 1 in the direction of MAA 2. Let $G_{\mathrm{r\,2\,MAX}}$ be the maximum reached gain, and $A_{\mathrm{r\,2}}$ the reached effective area, of MAA 2 in the direction of MAA 1. It follows from the definitions that:

$$G_{\mathrm{AT\,MAX}} \leqslant \frac{G_{\mathrm{r\,1\,MAX}}\,A_{\mathrm{r\,2}}}{4\pi\,d^2} \tag{173}$$

and

$$G_{\mathrm{BT\,MAX}} \leqslant \frac{G_{\mathrm{r\,2\,MAX}}\,A_{\mathrm{r\,1}}}{4\pi\,d^2}\,. \tag{174}$$

Let $G_{\mathrm{r\,1}}$ be the reached gain for a specified nonzero excitation, and $A_{\mathrm{pr\,1}}(\mathbf{u}_{\mathrm{pol}})$ a partial reached effective area, of MAA 1 in the direction of MAA 2. Let $G_{\mathrm{r\,2}}$ be the reached gain for a specified nonzero excitation, and $A_{\mathrm{pr\,2}}(\mathbf{u}_{\mathrm{pol}})$ a partial reached effective area, of MAA 2 in the direction of MAA 1.

If we assume that there exists an excitation of MAA 1 for which $G_{\mathrm{r\,1}} = G_{\mathrm{r\,1\,MAX}}$ in the direction of MAA 2, this excitation producing, at a large distance in the direction of MAA 2, a uniform incident plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,1}}$ such that $A_{\mathrm{r\,2}} = A_{\mathrm{pr\,2}}(\mathbf{u}_{\mathrm{pol\,1}})$, then:

(a) for this excitation, we have

$$G_{\mathrm{AT}} = \frac{G_{\mathrm{r\,1\,MAX}}\,A_{\mathrm{r\,2}}}{4\pi\,d^2}\,; \tag{175}$$

(b) it follows from (173) and (175) that

$$G_{\mathrm{AT\,MAX}} = \frac{G_{\mathrm{r\,1\,MAX}}\,A_{\mathrm{r\,2}}}{4\pi\,d^2}\,; \tag{176}$$

(c) if MAA 2 is reciprocal and $\mathbf{Z}_{S2}$ is symmetric, it follows from (111) and (176) that

$$G_{\mathrm{AT\,MAX}} = G_{\mathrm{r\,1\,MAX}}\,G_{\mathrm{r\,2\,MAX}}\left(\frac{\lambda}{4\pi\,d}\right)^2\,; \tag{177}$$

and

(d) if MAA 1 is reciprocal and $\mathbf{Z}_{S1}$ is symmetric, it follows from (111) and (176) that

$$G_{\mathrm{AT\,MAX}} = \frac{A_{\mathrm{r\,1}}\,A_{\mathrm{r\,2}}}{d^2\,\lambda^2}\,. \tag{178}$$

Likewise, if there exists an excitation of MAA 2 for which $G_{\mathrm{r\,2}} = G_{\mathrm{r\,2\,MAX}}$ in the direction of MAA 1, this excitation producing, at a large distance in the direction of MAA 1, a uniform incident plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,2}}$ such that $A_{\mathrm{r\,1}} = A_{\mathrm{pr\,1}}(\mathbf{u}_{\mathrm{pol\,2}})$, then:

(e) for this excitation, we have

$$G_{\mathrm{BT}} = \frac{G_{\mathrm{r\,2\,MAX}}\,A_{\mathrm{r\,1}}}{4\pi\,d^2}\,; \tag{179}$$

(f) it follows from (174) and (179) that

$$G_{\mathrm{BT\,MAX}} = \frac{G_{\mathrm{r\,2\,MAX}}\,A_{\mathrm{r\,1}}}{4\pi\,d^2}\,; \tag{180}$$

(g) if MAA 1 is reciprocal and $\mathbf{Z}_{S1}$ is symmetric, it follows from (111) and (180) that

$$G_{\mathrm{BT\,MAX}} = G_{\mathrm{r\,1\,MAX}}\,G_{\mathrm{r\,2\,MAX}}\left(\frac{\lambda}{4\pi\,d}\right)^2\,; \tag{181}$$

and

(h) if MAA 2 is reciprocal and $\mathbf{Z}_{S2}$ is symmetric, it follows from (111) and (180) that

$$G_{\mathrm{BT\,MAX}} = \frac{A_{\mathrm{r\,1}}\,A_{\mathrm{r\,2}}}{d^2\,\lambda^2}\,. \tag{182}$$

We observe that we can also assert logical converses of the statements (b) and (f), as follows:

(i) if (176) is satisfied, there exists an excitation of MAA 1 such that (175) is satisfied, this excitation being necessarily such that $G_{\mathrm{r\,1}} = G_{\mathrm{r\,1\,MAX}}$ in the direction of MAA 2, and such that it produces, at a large distance in the direction of MAA 2, an incident uniform plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,1}}$ such that $A_{\mathrm{r\,2}} = A_{\mathrm{pr\,2}}(\mathbf{u}_{\mathrm{pol\,1}})$; and

(j) if (180) is satisfied, there exists an excitation of MAA 2 such that (179) is satisfied, this excitation being necessarily such that $G_{\mathrm{r\,2}} = G_{\mathrm{r\,2\,MAX}}$ in the direction of MAA 1, and such that it produces, at a large distance in the direction of MAA 1, an incident uniform plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,2}}$ such that $A_{\mathrm{r\,1}} = A_{\mathrm{pr\,1}}(\mathbf{u}_{\mathrm{pol\,2}})$.

Consequently, if MAA 1 and MAA 2 are reciprocal and if $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric, we can assert that the statement $\mathcal{S}_3$ = "there exists an excitation of MAA 1 for which $G_{\mathrm{r\,1}} = G_{\mathrm{r\,1\,MAX}}$ in the direction of MAA 2, this excitation producing, at a large distance in the direction of MAA 2, a uniform incident plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,1}}$ such that $A_{\mathrm{r\,2}} = A_{\mathrm{pr\,2}}(\mathbf{u}_{\mathrm{pol\,1}})$" and the statement $\mathcal{S}_4$ = "there exists an excitation of MAA 2 for which $G_{\mathrm{r\,2}} = G_{\mathrm{r\,2\,MAX}}$ in the direction of MAA 1, this excitation producing, at a large distance in the direction of MAA 1, a uniform incident plane wave of polarization vector $\mathbf{u}_{\mathrm{pol\,2}}$ such that $A_{\mathrm{r\,1}} = A_{\mathrm{pr\,1}}(\mathbf{u}_{\mathrm{pol\,2}})$" are equivalent.

Proof: To show this, we can for instance assume that $\mathcal{S}_3$ is true. Thus, (177) is true because MAA 2 is reciprocal and $\mathbf{Z}_{S2}$ is symmetric. Since MAA 1, MAA 2 and the medium surrounding them are reciprocal, and since $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric, we can use Theorem 4 of [5] or Theorem 15 of [6] to assert that $G_{\mathrm{AT\,MAX}} = G_{\mathrm{BT\,MAX}}$, so that (177) entails (181). Using (111), we find that (180) is true. Using (j), we find that $\mathcal{S}_4$ is true. □

The transmission formulas (176)–(178) subject to the condition $\mathcal{S}_3$, the transmission formulas (180)–(182) subject to the condition $\mathcal{S}_4$, and the equivalence between $\mathcal{S}_3$ and $\mathcal{S}_4$ if MAA 1 and MAA 2 are reciprocal and if $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric, form a complete set of new transmission formulas.

TABLE 4. Results not based on the reciprocity of the MAA.

Equality	No.
$G_{\mathrm{pa\,MEA}}(\mathbf{u}_{\mathrm{pol}}) = \frac{G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})}{N}$	(52)
$(N > 2) \Longrightarrow (G_{\mathrm{a\,MIN}} = 0)$	(70)
$(N = 1) \Rightarrow \left(A_{\mathrm{aeq\,MEA}} = \frac{A_{\mathrm{a}}}{2} \text{ and } A_{\mathrm{aeq\,MIN}} = 0\right)$	(77)
$G_{\mathrm{pr\,MEA}}(\mathbf{u}_{\mathrm{pol}}) = \frac{G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})}{N}$	(99)
$(N > 2) \Longrightarrow (G_{\mathrm{r\,MIN}} = 0)$	(114)
$(N = 1) \Rightarrow \left(A_{\mathrm{req\,MEA}} = \frac{A_{\mathrm{r}}}{2} \text{ and } A_{\mathrm{req\,MIN}} = 0\right)$	(119)
$A_{\mathrm{aeq\,MEA}} = \langle A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}})\rangle_{f_{\mathrm{U}}}$	(155)
$A_{\mathrm{req\,MEA}} = \langle A_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}})\rangle_{f_{\mathrm{U}}}$	(156)

XV. CONCLUSION

We have named and carefully defined the parameters for emission listed in Table 1 (for excitation-dependent parameters) and Table 2 (for excitation-independent parameters), and the parameters for reception listed in Table 3. We have also: established how these parameters can be computed; proven the results listed in Table 4, which are not related to a possible reciprocity of the MAA; and proven the results listed in Table 5, which are based on the reciprocity of the MAA.

It is interesting to note that we had to introduce more parameters for emission than for reception.

We have used the parameters for emission and reception to obtain: new generalizations of the Friis transmission formula, which are quite different from the ones proposed in [6, Sec. XIII] and [17]–[19]; and other new transmission formulas. These formulas and the results listed in Table 5 justify the definitions of the parameters and prove that the difficulties mentioned in Section I have been overcome.

We acknowledge that [23] and several of the references listed in [19] are relevant to the subject matter of this article, but we have not tried to compare our definitions and results to others. It is nevertheless worth noting that we have ignored the "partial realized gain" and "realized gain" defined in [2] for a single-port antenna, the reference power used to obtain these parameters being an incident power, presumably determined for a specified reference resistance r_0. However, these "partial realized gain" and "realized gain" are encompassed in our more general concept of "partial reached gain" and "reached gain", for which the reference power is an available power from a generator having, in the special case of a single-port antenna, any impedance of positive real part.

In Section III.A, we have assumed (for the convenience of presentation) that "the MAA is passive and may be used for emission and reception at ω". However, if we now consider an LTI MAA that does not satisfy this requirement but is stable in the relevant configuration and complies with our other assumptions, we see that: the parameters listed in Table 1 and Table 2 may be defined if this MAA can be used for emission at ω; and the parameters listed in Table 3 may be defined if this MAA can be used for reception at ω.

TABLE 5. Results based on the reciprocity of the MAA.

Equality	No.
$A_{\mathrm{pa}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\lambda^2}{4\pi} G_{\mathrm{pa\,MAX}}(\mathbf{u}_{\mathrm{pol}})$	(42)
$A_{\mathrm{a}} = \frac{\lambda^2}{4\pi} G_{\mathrm{a\,MAX}}$	(65)
$A_{\mathrm{aeq\,MEA}} = \frac{N\lambda^2}{8\pi} G_{\mathrm{a\,MEA}}$	(81)
$(N = 2) \Longrightarrow \left(A_{\mathrm{aeq\,MIN}} = \frac{\lambda^2}{4\pi} G_{\mathrm{a\,MIN}}\right)$	(82)
$A_{\mathrm{pr}}(\mathbf{u}_{\mathrm{pol}}) = \frac{\lambda^2}{4\pi} G_{\mathrm{pr\,MAX}}(\mathbf{u}_{\mathrm{pol}})$	(91)
$A_{\mathrm{r}} = \frac{\lambda^2}{4\pi} G_{\mathrm{r\,MAX}}$	(111)
$A_{\mathrm{req\,MEA}} = \frac{N\lambda^2}{8\pi} G_{\mathrm{r\,MEA}}$	(123)
$(N = 2) \Longrightarrow \left(A_{\mathrm{req\,MIN}} = \frac{\lambda^2}{4\pi} G_{\mathrm{r\,MIN}}\right)$	(124)

APPENDIX A

In a complex vector space E, let f be a positive definite hermitian sesquililear form $f : E \times E \to \mathbb{C}$ [24, Ch. 3], [25, Sec. VI.4]. It may be selected as the inner product of E [24, Ch. 4], [25, Ch. VII]. In this case, for any $\mathbf{x} \in E$ and any $\mathbf{y} \in E$, the complex number $f(\mathbf{x}, \mathbf{y})$ may be denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, and, for any $\mathbf{x}' \in E$, $\mathbf{y}' \in E$ and $\alpha \in \mathbb{C}$, we have:

- (*i*) $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$;
- (*ii*) $\langle \mathbf{x}, \mathbf{y} + \mathbf{y}' \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y}' \rangle$;
- (*iii*) $\langle \alpha\, \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;
- (*iv*) $\langle \mathbf{x}, \alpha\, \mathbf{y} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$;
- (*v*) $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$;
- (*vi*) $\langle \mathbf{x}, \mathbf{x} \rangle \in \mathbb{R}$ and $\langle \mathbf{x}, \mathbf{x} \rangle \geqslant 0$; and
- (*vii*) $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

For this inner product and a positive integer p, an orthonormal family $(\mathbf{u}_1, \ldots, \mathbf{u}_p)$ of vectors of E is by definition such that $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{i,j}$, where i and j lie in $\{1, \ldots, p\}$ and $\delta_{i,j}$ is the Kronecker delta.

We now assume that E is of finite dimension ν, so that (E, f) is called a hermitian space [24, Ch. 4]. Let $(\mathbf{u}_1, \ldots, \mathbf{u}_\nu)$ be an orthonormal basis of this hermitian space. For any $\mathbf{x} \in E$ and any $\mathbf{y} \in E$, let $\mathbf{X}$ be the column vector of the coordinates of $\mathbf{x}$ in this basis, and $\mathbf{Y}$ be the column vector of the coordinates of $\mathbf{y}$ in this basis. By (*i*)–(*v*) we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{Y}^* \mathbf{X} \,. \tag{183}$$

Thus, $\mathbf{Y}^*\mathbf{X}$ does not depend on the choice of the orthonormal basis of (E, f). It also follows that the selection of a hermitian sesquililear form as an inner product of E is equivalent to the selection of a basis of E as an orthonormal basis.

In Section III, we have defined the real vector space $\mathbb{E}$ and the complex vector space $\mathbb{E}^{\mathrm{c}}$. The latter is referred to as the complexified of the real vector space $\mathbb{E}$ [24, Sec. 4.3], [25, Sec. VI.2]. In electromagnetics, it is customary to use a symmetric bilinear form $g : \mathbb{E}^{\mathrm{c}} \times \mathbb{E}^{\mathrm{c}} \to \mathbb{C}$ [24, Ch. 1], [25, Ch. II], such that, for any $\mathbf{x} \in \mathbb{E}^c$ and any $\mathbf{y} \in \mathbb{E}^c$, the complex number $g(\mathbf{x}, \mathbf{y})$ is denoted by $\mathbf{x} \cdot \mathbf{y}$, and, for any $\mathbf{x}' \in \mathbb{E}^c$, $\mathbf{y}' \in \mathbb{E}^c$ and $\alpha \in \mathbb{C}$, we have:

(*viii*) $(\mathbf{x} + \mathbf{x}') \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x}' \cdot \mathbf{y}$;
(*ix*) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{y}') = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}'$;
(*x*) $(\alpha\, \mathbf{x}) \cdot \mathbf{y} = \alpha(\mathbf{x} \cdot \mathbf{y})$;
(*xi*) $\mathbf{x} \cdot (\alpha\, \mathbf{y}) = \alpha(\mathbf{x} \cdot \mathbf{y})$;
(*xii*) $\mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{y}$;
(*xiii*) if $\mathbf{x} \in \mathbb{E}$, then $\mathbf{x} \cdot \mathbf{x} \in \mathbb{R}$ and $\mathbf{x} \cdot \mathbf{x} \geqslant 0$; and
(*xiv*) if $\mathbf{x} \in \mathbb{E}$, then $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

It follows from (*viii*)–(*xiv*) that this symmetric bilinear form is a scalar product in $\mathbb{E}$, so that $(\mathbb{E}, g)$ is an euclidean space [24, Ch. 2]. Let $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be an orthonormal basis of this euclidean space. For any $\mathbf{x}$ and $\mathbf{y}$ lying in $\mathbb{E}^c$, let $\mathbf{X}$ be the real column vector of the coordinates of $\mathbf{x}$ in this basis, and $\mathbf{Y}$ be the real column vector of the coordinates of $\mathbf{y}$ in this basis. By (*viii*)–(*xii*) we have

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{Y}^{\mathrm{T}} \mathbf{X} \,. \tag{184}$$

Thus, $\mathbf{Y}^{\mathrm{T}} \mathbf{X}$ does not depend on the choice of the orthonormal basis $\mathcal{B}$ of $(\mathbb{E}, g)$.

For any $\mathbf{x} \in \mathbb{E}^c$, let $\mathbf{X}$ be the column vector of the coordinates of $\mathbf{x}$ in $\mathcal{B}$. Recall that $\mathcal{B}$ was defined as an orthonormal basis of $(\mathbb{E}, g)$. Let $\overline{\mathbf{x}} \in \mathbb{E}^c$ be such that the column vector of the coordinates of $\overline{\mathbf{x}}$ in $\mathcal{B}$ is $\overline{\mathbf{X}}$. The vector $\overline{\mathbf{x}}$ is called the conjugate of $\mathbf{x}$. Let $\mathcal{B}' = (\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3)$ be a basis of $\mathbb{E}^c$, in which the column vector of the coordinates of $\mathbf{x}$ is denoted by $\mathbf{X}'$, and the column vector of the coordinates of $\overline{\mathbf{x}}$ is denoted by $\mathbf{M}'(\overline{\mathbf{x}})$. Let $\mathbf{C}$ be the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$. We have $\mathbf{X} = \mathbf{C}\mathbf{X}'$ and $\overline{\mathbf{X}} = \mathbf{C}\mathbf{M}'(\overline{\mathbf{x}})$, so that

$$\mathbf{M}'(\overline{\mathbf{x}}) = \mathbf{C}^{-1}\, \overline{\mathbf{C}}\; \overline{\mathbf{X}'} \,. \tag{185}$$

It follows that the proposition

$$\mathcal{H} = \left(\text{for any } \mathbf{x} \in \mathbb{E}^c,\ \mathbf{M}'(\overline{\mathbf{x}}) = \overline{\mathbf{X}'}\,\right) \tag{186}$$

is true if and only if $\mathbf{C}$ is a real matrix, that is to say, if and only if $\mathcal{B}'$ is a basis of $\mathbb{E}$.

Consequently, in any basis of $\mathbb{E}$ the column vector of the coordinates of $\overline{\mathbf{x}}$ is the conjugate of the column vector of the coordinates of $\mathbf{x}$. It follows that the conjugate of a vector does not depend on the basis of $\mathbb{E}$ used to define it.

The symmetric bilinear form g does not satisfy (*iv*)–(*vii*), so that it cannot be used as an inner product in $\mathbb{E}^c$. However, it is used to select a hermitian sesquililear form f as an inner product of $\mathbb{E}^c$, by deciding that the orthonormal basis $\mathcal{B}$ of $(\mathbb{E}, g)$ is an orthonormal basis of $(\mathbb{E}^c, f)$.

Thus, for any $\mathbf{x}$ and $\mathbf{y}$ lying in $\mathbb{E}^c$, if $\mathbf{X}$ and $\mathbf{Y}$ are the column vectors of the coordinates of $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{B}$, respectively, it follows from (183) that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{Y}^* \mathbf{X}$, and from (184) that $\mathbf{x} \cdot \overline{\mathbf{y}} = \overline{\mathbf{y}} \cdot \mathbf{x} = \mathbf{Y}^* \mathbf{X}$. This leads us to the vector equation

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \overline{\mathbf{y}} = \overline{\mathbf{y}} \cdot \mathbf{x} \,. \tag{187}$$

We note that, for any $\mathbf{x} \in \mathbb{E}$, we have $\overline{\mathbf{x}} = \mathbf{x}$. Thus, it follows from (187) that any basis of $\mathbb{E}$ is an orthonormal basis of the euclidean space $(\mathbb{E}, g)$ if and only if it is an orthonormal basis of $(\mathbb{E}^c, f)$. Consequently, the inner product in $\mathbb{E}^c$ does not depend on the orthonormal basis $\mathcal{B}$ of $(\mathbb{E}, g)$ used to define it. It also follows that the wording "orthonormal basis of $\mathbb{E}$" is not ambiguous.

An arbitrary orthonormal basis of $\mathbb{E}$ being chosen, for any $\mathbf{v}$ and $\mathbf{w}$ lying in $\mathbb{E}^c$, the coordinates of $\mathbf{v}$ and $\mathbf{w}$ in this basis are now denoted by $\underset{\sim}{\mathbf{v}}$ and $\underset{\sim}{\mathbf{w}}$, respectively. It follows from (183) and (187) that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \underset{\sim}{\mathbf{w}}^* \underset{\sim}{\mathbf{v}} = \mathbf{v} \cdot \overline{\mathbf{w}} = \overline{\mathbf{w}} \cdot \mathbf{v} \,. \tag{188}$$

Orthogonality in $\mathbb{E}^c$ and the norm in $\mathbb{E}^c$ are defined with respect to $(\mathbb{E}^c, f)$. Accordingly, $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \overline{\mathbf{v}}} = \sqrt{\overline{\mathbf{v}} \cdot \mathbf{v}}$, and $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \overline{\mathbf{w}} = \overline{\mathbf{w}} \cdot \mathbf{v} = 0$.

The bilinear form g is used in (187) and (188) to express the inner product by utilizing a conjugate vector. We must keep in mind that the bilinear form g intrinsically does not behave like an inner product in $\mathbb{E}^c$. For instance, if we consider $\underset{\sim}{\mathbf{v}} = (1, j, 0)^{\mathrm{T}}$, we have $\mathbf{v} \cdot \mathbf{v} = 0$. The bilinear form is nevertheless used in the main body of this article, for instance in the formula (13) defining vector effective heights for reception, and in some formulas containing a polarization vector, such as (33) and (83).

By the Cauchy-Schwarz inequality for positive definite hermitian sesquililear forms [24, Sec. 3.3.2], [25, Sec. VI.8], for any $\mathbf{v} \in \mathbb{E}^c$ and any $\mathbf{w} \in \mathbb{E}^c$, we have

$$|\langle \mathbf{v}, \mathbf{w} \rangle|^2 \leqslant \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle \,, \tag{189}$$

with equality if and only if there exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha \mathbf{v} + \beta \mathbf{w} = \mathbf{0}$. It follows that

$$|\mathbf{v} \cdot \mathbf{w}| \leqslant ||\mathbf{v}||\ ||\mathbf{w}|| \,, \tag{190}$$

with equality if and only if there exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha \mathbf{v} + \beta\, \overline{\mathbf{w}} = \mathbf{0}$. This result can be applied to the polarization mismatch factor of the RAA, defined by (25) for $\mathbf{E}_{\mathrm{AR}} \neq \mathbf{0}$ and $\mathbf{h}_{\mathrm{R1}} \neq \mathbf{0}$, to assert that

$$e_{\mathrm{pol}} \leqslant 1 \,, \tag{191}$$

with equality if and only if there exists $\mu \in \mathbb{C}$ such that we have $\mathbf{E}_{\mathrm{AR}} = \mu\, \overline{\mathbf{h}_{\mathrm{R1}}}$.

APPENDIX B

This Appendix B provides clarifications about the assumption of Section III.A regarding the distance r_{R}, and a proof of the fact that the premises of Theorem 29 of [6] are satisfied. To this end, we refer to the MAA, the RAA and said lossless medium lying around the MAA and the RAA as the "device under study" (DUS) having 2 sets of ports, referred to as port set 1 and port set 2. Port set 1 consists of the N ports of the MAA, and port set 2 consists of the n ports of the RAA. We introduce a series-augmented multiport, as defined in [5, Sec. II], composed of the DUS (as original multiport), of an N-port load of impedance matrix $\mathbf{Z}_{\mathrm{G}}$ connected in series with port set 1, and of an n-port load of impedance matrix $\mathbf{Z}_{\mathrm{R}}^*$ connected in series with port set 2. Here, the impedance matrix of the added multiport is

$$\mathbf{Z}_{\mathrm{ADD}} = \begin{pmatrix} \mathbf{Z}_{\mathrm{G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_{\mathrm{R}}^* \end{pmatrix} . \tag{192}$$

We have assumed that $H(\mathbf{Z}_\mathrm{G})$ and $H(\mathbf{Z}_\mathrm{R})$ are positive definite (see sections III.A and III.B), so that $H(\mathbf{Z}_\mathrm{ADD})$ is positive definite. By Theorem 2 of [5], the series-augmented multiport has an admittance matrix $\mathbf{Y}_\mathrm{SAM}$ such that $H(\mathbf{Y}_\mathrm{SAM})$ is positive semidefinite. The matrix $\mathbf{Y}_\mathrm{SAM}$ is of size $(N+n)$ by $(N+n)$. It may be partitioned into four submatrices, $\mathbf{Y}_\mathrm{SAM11}$ of size N by N, $\mathbf{Y}_\mathrm{SAM12}$ of size N by n, $\mathbf{Y}_\mathrm{SAM21}$ of size n by N and $\mathbf{Y}_\mathrm{SAM22}$ of size n by n, which are such that

$$\mathbf{Y}_\mathrm{SAM} = \begin{pmatrix} \mathbf{Y}_\mathrm{SAM11} & \mathbf{Y}_\mathrm{SAM12} \\ \mathbf{Y}_\mathrm{SAM21} & \mathbf{Y}_\mathrm{SAM22} \end{pmatrix} . \tag{193}$$

$\mathbf{Z}_\mathrm{R}$ is symmetric since the RAA is reciprocal. Thus, by Theorem 2 of [5], if $\mathbf{Z}_\mathrm{G}$ is symmetric and the MAA is reciprocal, then $\mathbf{Y}_\mathrm{SAM}$ is symmetric.

It follows from our assumptions that the DUS has an impedance matrix $\mathbf{Z}$, of size $(N+n)$ by $(N+n)$, which is such that

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}'_\mathrm{A} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}'_\mathrm{R} \end{pmatrix} , \tag{194}$$

where $\mathbf{Z}_{12}$ is a matrix of size N by n and $\mathbf{Z}_{21}$ is a matrix of size n by N. It follows from (14) and (31) that we have

$$\mathbf{Z}_{12} = j\eta \frac{k\, e^{-jkr_\mathrm{R}}}{4\pi r_\mathrm{R}} (\mathbf{h}_\mathrm{B}^\mathrm{T}\mathbf{h}_\mathrm{R} + \Delta_{12}) , \tag{195}$$

and from (11) and (20) that we have

$$\mathbf{Z}_{21} = j\eta \frac{k\, e^{-jkr_\mathrm{R}}}{4\pi r_\mathrm{R}} (\mathbf{h}_\mathrm{R}^\mathrm{T}\mathbf{h}_\mathrm{A} + \Delta_{21}) , \tag{196}$$

where $\mathbf{h}_\mathrm{A}$ and $\mathbf{h}_\mathrm{B}$ are considered in the direction $\theta = \theta_\mathrm{R}$ and $\varphi = \varphi_\mathrm{R}$, where $\mathbf{h}_\mathrm{R}$ is considered in the opposite direction $\theta' = \pi/2$ and $\varphi' = \pi$, where Δ_{12} is a matrix of size N by n having the dimensions of area and representing a small correction to $\mathbf{h}_\mathrm{B}^\mathrm{T}\mathbf{h}_\mathrm{R}$ that accounts for the finite distance between the MAA and the RAA, and where Δ_{21} is a matrix of size n by N having the dimensions of area and representing a small correction to $\mathbf{h}_\mathrm{R}^\mathrm{T}\mathbf{h}_\mathrm{A}$ that accounts for the finite distance between the MAA and the RAA.

If r_R is considered as a variable, it follows from our definitions that all entries of the matrices $\mathbf{Z}'_\mathrm{A} - \mathbf{Z}_\mathrm{A}$, $\mathbf{Z}'_\mathrm{R} - \mathbf{Z}_\mathrm{R}$, Δ_{12} and Δ_{21} tend to zero as r_R becomes large.

Since $H(\mathbf{Z}_\mathrm{A})$ and $H(\mathbf{Z}_\mathrm{R})$ are positive definite, each of them has a smallest eigenvalue which is positive, by [7, Sec. 7.2.1]. The eigenvalues of a square complex matrix depending continuously on its entries [8, Appendix D], the smallest eigenvalues of $H(\mathbf{Z}'_\mathrm{A})$ and $H(\mathbf{Z}'_\mathrm{R})$ are also positive if r_R is sufficiently large. According to the assumption of Section III.A regarding the distance r_R, this is indeed the case. Consequently, we can use [7, Sec. 7.2.1] again to assert that $H(\mathbf{Z}'_\mathrm{A})$ and $H(\mathbf{Z}'_\mathrm{R})$ are positive definite.

It follows that $H(\mathbf{Z}'_\mathrm{R} + \mathbf{Z}^*_\mathrm{R})$ and $H(\mathbf{Z}'_\mathrm{A} + \mathbf{Z}_\mathrm{G})$ are positive definite. By Lemma 1 of [5], it follows that $\mathbf{Z}'_\mathrm{R} + \mathbf{Z}^*_\mathrm{R}$ and $\mathbf{Z}'_\mathrm{A} + \mathbf{Z}_\mathrm{G}$ are invertible. This fact and the structure of the series-augmented multiport lead us to

$$\mathbf{Z}''_\mathrm{A} = \mathbf{Z}'_\mathrm{A} - \mathbf{Z}_{12}(\mathbf{Z}'_\mathrm{R} + \mathbf{Z}^*_\mathrm{R})^{-1}\mathbf{Z}_{21} \tag{197}$$

and

$$\mathbf{Z}''_\mathrm{R} = \mathbf{Z}'_\mathrm{R} - \mathbf{Z}_{21}(\mathbf{Z}'_\mathrm{A} + \mathbf{Z}_\mathrm{G})^{-1}\mathbf{Z}_{12} . \tag{198}$$

We have assumed that the RAA is reciprocal and $\mathbf{Z}_\mathrm{R}$ is symmetric. If we further assume that the MAA is reciprocal and $\mathbf{Z}_\mathrm{G}$ is symmetric, then $\mathbf{Z}_\mathrm{A}$, $\mathbf{Z}'_\mathrm{A}$, $\mathbf{Z}'_\mathrm{R}$, $\mathbf{Z}_\mathrm{ADD}$ and $\mathbf{Z}$ are symmetric, and $\mathbf{h}_\mathrm{A} = \mathbf{h}_\mathrm{B}$. In this case, we consequently have $\Delta_{21} = \Delta_{12}^\mathrm{T}$.

Since $\mathbf{Z}$ exists, it follows from Corollary 2 of [5] that: $\mathbf{Y}_\mathrm{SAM}$ is invertible;

$$\mathbf{Y}_\mathrm{SAM}^{-1} = \mathbf{Z} + \mathbf{Z}_\mathrm{ADD} ; \tag{199}$$

and, if $\mathbf{Z}_\mathrm{ADD}$ is symmetric, $\mathbf{Y}_\mathrm{SAM}$ is symmetric if and only if $\mathbf{Z}$ is symmetric.

$\mathbf{Z}$ being the impedance matrix of a passive system, $H(\mathbf{Z})$ is positive semidefinite. $H(\mathbf{Z}_\mathrm{ADD})$ being positive definite, it follows from (199) that $H(\mathbf{Y}_\mathrm{SAM}^{-1})$ is positive definite. Thus, $H(\mathbf{Y}_\mathrm{SAM})$ is positive definite by Lemma 1 of [5]. By [7, Sec. 7.1.2], it follows that $H(\mathbf{Y}_\mathrm{SAM11})$ is positive definite and $H(\mathbf{Y}_\mathrm{SAM22})$ is positive definite.

Using again Lemma 1 of [5], we find that $\mathbf{Y}_\mathrm{SAM11}$ and $\mathbf{Y}_\mathrm{SAM22}$ are invertible, so that propositions $\mathcal{P}_1$ and $\mathcal{P}_2$ of [6], defined in [6, Sec. VI.D], are true.

Since $\mathbf{Y}_\mathrm{SAM11}$ and $\mathbf{Y}_\mathrm{SAM22}$ are invertible, it follows from (197)–(198) and the structure of the series-augmented multiport, or from [7, Sec. 0.7.3], (192), (194) and (199), that

$$\mathbf{Y}_\mathrm{SAM11}^{-1} = \mathbf{Z}'_\mathrm{A} + \mathbf{Z}_\mathrm{G} - \mathbf{Z}_{12}(\mathbf{Z}'_\mathrm{R} + \mathbf{Z}^*_\mathrm{R})^{-1}\mathbf{Z}_{21} \tag{200}$$

and

$$\mathbf{Y}_\mathrm{SAM22}^{-1} = \mathbf{Z}'_\mathrm{R} + \mathbf{Z}^*_\mathrm{R} - \mathbf{Z}_{21}(\mathbf{Z}'_\mathrm{A} + \mathbf{Z}_\mathrm{G})^{-1}\mathbf{Z}_{12} . \tag{201}$$

Consequently, we have

$$H(\mathbf{Y}_\mathrm{SAM11}^{-1} - \mathbf{Z}_\mathrm{G}) = H(\mathbf{Z}'_\mathrm{A}) - H(\mathbf{Z}_{12}(\mathbf{Z}'_\mathrm{R} + \mathbf{Z}^*_\mathrm{R})^{-1}\mathbf{Z}_{21}) \tag{202}$$

and

$$H(\mathbf{Y}_\mathrm{SAM22}^{-1} - \mathbf{Z}^*_\mathrm{R}) = H(\mathbf{Z}'_\mathrm{R}) - H(\mathbf{Z}_{21}(\mathbf{Z}'_\mathrm{A} + \mathbf{Z}_\mathrm{G})^{-1}\mathbf{Z}_{12}) . \tag{203}$$

Since $H(\mathbf{Z}'_\mathrm{A})$ and $H(\mathbf{Z}'_\mathrm{R})$ are positive definite, each of them has a smallest eigenvalue which is positive. The eigenvalues of a square complex matrix depending continuously on its entries, it follows from (195)–(196) and (202)–(203) that the least eigenvalue of $H(\mathbf{Y}_\mathrm{SAM11}^{-1} - \mathbf{Z}_\mathrm{G})$ and the least eigenvalue of $H(\mathbf{Y}_\mathrm{SAM22}^{-1} - \mathbf{Z}^*_\mathrm{R})$ are positive if r_R is sufficiently large. According to the assumption of Section III.A regarding the distance r_R, this is indeed the case. Thus, $H(\mathbf{Y}_\mathrm{SAM11}^{-1} - \mathbf{Z}_\mathrm{G})$ and $H(\mathbf{Y}_\mathrm{SAM22}^{-1} - \mathbf{Z}^*_\mathrm{R})$ are positive definite. It follows that propositions $\mathcal{P}_3$ and $\mathcal{P}_4$ of [6], defined in [6, Sec. VI.D], are true.

In (i) of Theorem 29 of [6], the conditions for the applicability of (298)-(302) include "if the DUS and both loads are reciprocal devices".

In Section IV above, it follows from (39), (41), (44) and (47)–(48) that $A_\mathrm{pa}(\mathbf{u}_\mathrm{pol})$, g_BU1, $G_\mathrm{pa}(\mathbf{u}_\mathrm{pol})$ and g_AU1 do not depend on $\mathbf{Z}_\mathrm{G}$. Accordingly, it follows from the definitions of the MLR in Section III.E that the requirement "if the DUS and both loads are reciprocal devices" can be replaced with "if the MAA is reciprocal" in Section IV.

In Section V and Section VI above, it follows from (54), (60)–(61), (67)–(68) and (79) that $A_{\text{pa}}(\mathbf{u}_{\text{pol}})$, $g_{\text{BU2 pol}}$, G_{a} and g_{AU2} do not depend on $\mathbf{Z}_{\text{G}}$. Accordingly, it follows from the definitions of the MLR in Section III.E that the requirement "if the DUS and both loads are reciprocal devices" can be replaced with "if the MAA is reciprocal" in Section V and Section VI.

REFERENCES

[1] *IEC 60050-712, First Edition (1992-04), International Electrotechnical Vocabulary (IEV) - Part 712: Antennas*. Genève, Switzerland: IEC, 1992.

[2] *IEEE Std 145-2013, IEEE Standard for Definitions of Terms for Antennas*, New York, NY, USA: IEEE, 2014.

[3] F. Broydé and E. Clavelier, "About the Power Ratios Relevant to a Passive Linear Time-Invariant 2-Port," *Excem Research Papers in Electronics and Electromagnetics*, no. 5, doi: 10.5281/zenodo.6555682, May 2022.

[4] F. Broydé and E. Clavelier, "Some Results on Power in Passive Linear Time-Invariant Multiports, Part 2," *Excem Research Papers in Electronics and Electromagnetics*, no. 3, doi: 10.5281/zenodo.4683896, Apr. 2021.

[5] F. Broydé and E. Clavelier, "Some Results on Power in Passive Linear Time-Invariant Multiports, Part 1," *Excem Research Papers in Electronics and Electromagnetics*, no. 2, doi: 10.5281/zenodo.4419637, Jan. 2021.

[6] F. Broydé and E. Clavelier, "Some Results on Power in Passive Linear Time-Invariant Multiports, Part 3," *Excem Research Papers in Electronics and Electromagnetics*, no. 7, doi: 10.5281/zenodo.8124511, Jul. 2023.

[7] R.A. Horn and C.R. Johnson, *Matrix analysis*, 2nd ed., New York, NY, USA: Cambridge University Press, 2013.

[8] R.A. Horn and C.R. Johnson, *Matrix analysis*, New York, NY, USA: Cambridge University Press, 1985.

[9] F. Broydé and E. Clavelier, "The Radiation and Transducer Efficiencies of a Multiport Antenna Array," *Excem Research Papers in Electronics and Electromagnetics*, no. 4, doi: 10.5281/zenodo.5816837, Jan. 2022.

[10] E.C. Jordan and K.G. Balmain, *Electromagnetic Waves and Radiating Systems*, 2nd Edition, Englewood Cliffs, NJ, USA: Prentice-Hall, 1968.

[11] G. Sinclair, "The transmission and reflection of elliptically polarized waves," *Proc. IRE*, vol. 38, no. 2, pp. 148-151, Feb. 1950.

[12] G. Sinclair, "Correction to 'The transmission and reflection of elliptically polarized waves'," *Proc. IRE*, vol. 38, no. 10, p. 1216, Oct. 1950.

[13] R.E. Collin, *Antennas and Radiowave Propagation*, International Edition, New York, NY, USA: McGraw-Hill, 1985.

[14] S.J. Orfanidis, *Electromagnetic Waves and Antennas — vol. 2 — Antennas*, Sophocles J. Orfanidis, 2016.

[15] C.A. Balanis, *Antenna Theory*, 2nd ed., New York, NY, USA: John Wiley & Sons, 1997.

[16] E.W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, Boca Raton, FL, USA: CRC Press, 1999.

[17] F. Broydé, E. Clavelier, L. Jelinek, M. Capek and K.F. Warnick "Implementing Two Generalizations of the Friis Transmission Formula," *Excem Research Papers in Electronics and Electromagnetics*, no. 8, https://doi.org/10.5281/zenodo.10448398, Jan. 2024.

[18] L. Jelinek, M. Capek, E. Clavelier, K.F. Warnick and F. Broydé, "Generalized Friis transmission formula for multiport antenna arrays," *Proc. 2024 IEEE Int. Symp. on Antennas and Propagation and INC-USNC-URSI Radio Science Meeting*, Jul. 2024.

[19] K.F. Warnick, F. Broydé, L. Jelinek, M. Capek and E. Clavelier, "Generalized Friis transmission formula using active antenna available power and unnamed power gain," *IEEE Trans. Antennas Propag.*, vol. 72, no. 8, pp. 6321-6331, Aug. 2024.

[20] H.T. Friis, "A note on a simple transmission formula," *Proceedings of the I.R.E. and Waves and Electrons*, vol. 34, no. 5, pp. 254-256, May 1946.

[21] H.T. Friis, "Introduction to radio and radio antennas," *IEEE Spectrum*, vol. 8, no. 4, pp. 55-61, Apr. 1971.

[22] W.L. Stutzman and G.A. Thiele, *Antenna Theory and Design*, Third Edition, Hoboken, NJ, USA: John Wiley & Sons, 2013.

[23] M. Capek, L. Jelinek and M. Masek, "Finding optimal total active reflection coefficient and realized gain for multi-port lossy antennas," *IEEE Trans. Antennas Propag.*, vol. 69, no. 5, pp. 2481–2493, May 2021.

[24] E. Ramis, C. Deschamps and J. Odoux, *Cours de mathématiques spéciales — 2 — Algèbre et applications à la géométrie*, Paris, France: Masson, 1979.

[25] R. Deheuvels, *Formes quadratiques et groupes classiques*, Paris, France: Presses Universitaires de France, 1981.

FRÉDÉRIC BROYDÉ was born in France in 1960. He received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble (ENSIEG) and the Ph.D. in microwaves and microtechnologies from the Université des Sciences et Technologies de Lille (USTL).

He co-founded the Excem corporation in May 1988, a company providing engineering and research and development services. He is president of Excem since 1988. He is now also president and CTO of Eurexcem, a subsidiary of Excem. Most of his activity is allocated to engineering and research in electronics, radio, antennas, electromagnetic compatibility (EMC) and signal integrity.

Dr. Broydé is author or co-author of about 117 technical papers, and inventor or co-inventor of about 96 patent families, for which 75 patents of France and 49 patents of the USA have been granted. He is a Senior Member of the IEEE since 2001. He is a licensed radio amateur (F5OYE).

EVELYNE CLAVELIER was born in France in 1961. She received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble (ENSIEG).

She is co-founder of the Excem corporation, based in Maule, France, and she is currently CEO of Excem. She is also president of Tekcem, a company selling or licensing intellectual property rights to foster research. She is an active engineer and researcher.

Her current research areas are radio communications, antennas, antenna arrays, EMC and circuit theory.

Prior to starting Excem in 1988, she worked for Schneider Electrics (in Grenoble, France), STMicroelectronics (in Grenoble, France), and Signetics (in Mountain View, CA, USA).

Ms. Clavelier is the author or a co-author of about 98 technical papers. She is co-inventor of about 93 patent families. She is a Senior Member of the IEEE since 2002. She is a licensed radio amateur (F1PHQ).

•••

 Published by Excem. ISBN 978-2-909056-30-2
Excem — 12, chemin des Hauts de Clairefontaine — 78580 Maule — France